

Giant vortex state in fast rotating Bose-Einstein Condensates

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Model for a rotating Bose-Einstein Condensate: the Gross-Pitaevskii model

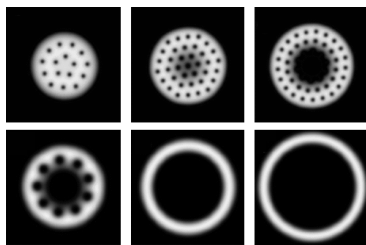
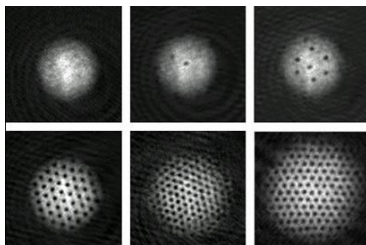
The state of the system is obtained minimizing the Gross-Pitaevskii energy functional:

$$\begin{aligned}\mathcal{E}^{\text{GP}}[\Psi] &= \int_{\mathbb{R}^2} d\mathbf{r} \left\{ \frac{1}{2} |\nabla \Psi|^2 + V(\mathbf{r}) |\Psi|^2 - \Omega_{\text{rot}} \Psi^* L \Psi + \frac{1}{\varepsilon^2} |\Psi|^4 \right\} = \\ &= \int_{\mathbb{R}^2} d\mathbf{r} \left\{ \frac{1}{2} |(\nabla - i\mathbf{A}_{\text{rot}}) \Psi|^2 + \left(V(\mathbf{r}) - \frac{1}{2} \Omega_{\text{rot}}^2 r^2 \right) |\Psi|^2 + \frac{1}{\varepsilon^2} |\Psi|^4 \right\}\end{aligned}$$

- $V = kr^s$, $s > 2$ is the trapping potential;
- $L = -i\partial_\theta$ angular momentum;
- $\mathbf{A}_{\text{rot}} = \Omega_{\text{rot}} \mathbf{r}^\perp$ vector potential;
- competition between V and the *centrifugal term*, different cases if V quadratic or more than quadratic

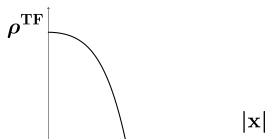
What happens if you increase the rotating speed Ω_{rot} ?:

- formation of quantized vortices in the condensate (related to [superfluidity properties](#));
- different shape of the condensate



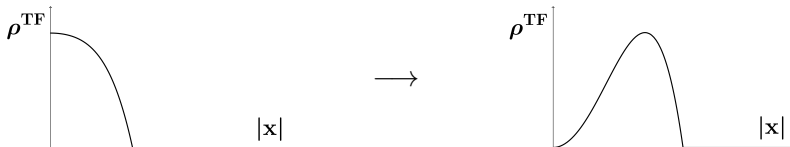
First regime: $\Omega_{\text{rot}} \ll \varepsilon^{-\frac{s-2}{s+2}}$

- in this regime the *mass distribution* of the condensate is well represented by the minimizer of a Thomas-Fermi functional and thus it is essentially supported around the origin;



- in particular while $\Omega_{\text{rot}} \ll \varepsilon^{\frac{4}{s+2}} |\log \varepsilon|$ the minimizer has no vortices at all;
- when Ω_{rot} pass a critical value $\Omega_1 \sim \varepsilon^{\frac{4}{s+2}} |\log \varepsilon|$ a **first vortex in the origin** is nucleated and when $\Omega_{\text{rot}} \gg \varepsilon^{\frac{4}{s+2}} |\log \varepsilon|$ the vortices are **uniformly distributed** in the whole plane $\mathbb{R}^2$ [first phase transition/**first critical speed**]

Second regime: $\Omega_{\text{rot}} \sim \varepsilon^{-\frac{s-2}{s+2}}$



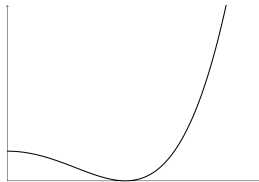
In this regime the Thomas-Fermi minimizer digs a hole in the origin, and this has the effect that the mass distribution of the condensate is exponentially small around the origin and becomes **essentially supported in an annulus far from the origin**; this corresponds to a **second critical speed** for the condensate of order $\Omega_2 \sim \varepsilon^{-\frac{s-2}{s+2}}$

Second regime: $\Omega_{\text{rot}} \sim \varepsilon^{-\frac{s-2}{s+2}}$

Scaling: the condensate becomes **essentially supported in an annulus** which radius goes to infinity; we rescale dimension such that the annulus remains at a fixed distance from the origin - that we choose to be 1

$$\mathcal{E}_{\Omega}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} d\mathbf{x} \left\{ \frac{1}{2} |(\nabla - i\mathbf{A}_{\Omega}) \psi|^2 + \Omega^2 W(\mathbf{x}) |\psi|^2 + \frac{1}{\varepsilon^2} |\psi|^4 \right\}$$

$$W(\mathbf{x}) = \frac{x^s - 1}{s} - \frac{x^2 - 1}{2}$$



Third regime: $\Omega \gg \varepsilon^{-1}$

- while $\varepsilon^{-1} \ll \Omega \ll \varepsilon^{-4}$ the mass distribution of the condensate is still close to the minimizer of another Thomas-Fermi functional, supported in an annulus of width $\sim (\varepsilon\Omega)^{-\frac{2}{3}}$ going to 0 as $\varepsilon \rightarrow 0$;



- moreover as long as $\Omega \ll \varepsilon^{-4}$ the minimizer has still vortices distributed in the whole $\mathbb{R}^2$;
- as soon as Ω pass a critical value $\Omega_3 \sim \varepsilon^{-4}$ the vortices in the annulus disappear: **third phase transition**, minimizer is in the so-called **Giant Vortex state**

For the first two critical speed the exact values are known:

$$\Omega_1 = \frac{\pi}{2} \left(\frac{2k(s+2)}{\pi s} \right)^{\frac{s}{s+2}} \varepsilon^{\frac{4}{s+2}} |\log \varepsilon|,$$

$$\Omega_2 = (2k)^{\frac{s}{s+2}} \left[\frac{4(s+2)}{\pi(s-2)} \right]^{\frac{s-2}{2(s+2)}} \varepsilon^{-\frac{s-2}{s+2}},$$

Theorem (M. Correggi, N. Rougerie, F. Pinsky, J. Yngvason, 2012)

It exists a constant Ω_c such that if Ω is of the form $\frac{\Omega_0}{\varepsilon^4}$ in the rescaled variables with Ω_0 fixed as $\varepsilon \rightarrow 0$ and $\Omega_0 > \Omega_c$ then **no minimizer has a zero in the annulus**. More precisely, exists a real radial strictly positive function f such that in the annulus we have that

$$|\psi^{\text{GP}}(\mathbf{x})| = f(x) (1 + o(1))$$

$$\Omega_c = ?$$

Giant Vortex phase

When $\Omega \gtrsim \varepsilon^{-4}$ any minimizer of the Gross-Pitaevskii energy functional behaves like a **real radial function that doesn't vanish in the annulus** times a phase factor of degree α around the origin, namely

$$\psi^{\text{GP}}(\mathbf{x}) \simeq f(\mathbf{x})e^{i\alpha\theta}$$

It is then natural to define the **Giant Vortex energy functional** as

$$\mathcal{E}_\alpha^{\text{gv}}[f] = \mathcal{E}_\Omega^{\text{GP}}[fe^{i\alpha\theta}]$$

$$E_\Omega^{\text{GP}} \leq E_\alpha^{\text{gv}} \quad \forall \alpha \implies E_\Omega^{\text{GP}} \leq \inf_\alpha E_\alpha^{\text{gv}} = E_{\alpha_*}^{\text{gv}}$$

$$\alpha_* = \Omega (1 + \mathcal{O}(\varepsilon^4))$$

Once you find an upper bound for the energy you may wonder whether it is possible to match it with a corresponding lower bound or not; here things get tricky. Using the critical value α_* found before and trying to get the lower bound a critical value for Ω_0 naturally arise

Main result (M. Correggi, D. D., 2015)

If $\Omega_0 \geq \Omega_c$, with Ω_c explicit, then

$$E^{\text{GP}} = E_{\alpha_*}^{\text{gv}} (1 + \mathcal{O}(\varepsilon^4))$$

and moreover no minimizer of the Gross-Pitaevskii energy functional has a zero in the annulus

- while $\Omega_0 \geq \Omega_c$ there are no vortices in the annulus so the phase transition actually happened and this means that $\Omega_3 \leq \frac{\Omega_c}{\epsilon^4}$;
- it is still an open problem to show that the value we just found is in fact the real critical speed for the condensate, i. e. that in fact $\Omega_3 = \frac{\Omega_c}{\epsilon^4}$, and to prove this means to show that for any $\Omega < \frac{\Omega_c}{\epsilon^4}$ there are vortices inside the annulus; the minimality argument for α_* gives us hope that the value we found is the critical one

Thanks for the attention!