

The Hamiltonian of the harmonic oscillator with an attractive δ' -interaction centred at the origin as approximated by the one with a triple of attractive δ -interactions

Sergio Albeverio^{1,2,3,4}, Silvestro Fassari^{2,5,6} and Fabio Rinaldi^{2,6,7}

¹Institut für Angewandte Mathematik, HCM, IZKS, BiBoS, Universität Bonn, Endenicherallee 60, D-53115 Bonn, Germany

²CERFIM, PO Box 1132, CH-6601 Locarno, Switzerland

³Centre Interfacultaire Bernoulli, EPFL, CH-1015 Lausanne, Switzerland

⁴Chair Professorship, Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, KSA

⁵ISR, Aeulistr. 10, CH-9470 Buchs, Switzerland

⁶Università degli Studi Guglielmo Marconi, Via Plinio 44, I-00193 Rome, Italy

⁷BIS Group of Institutions, Gagra-Moga, Punjab under Punjab Technical University, Punjab, India

E-mail: silvestro.fassari@isr.ch

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Abstract

In this note we provide an alternative way of defining the self-adjoint Hamiltonian of the harmonic oscillator perturbed by an attractive δ' -interaction, of strength β , centred at 0 (the bottom of the confining parabolic potential), that was rigorously defined in a previous paper by means of a “coupling constant renormalisation”. Here we get the Hamiltonian as a norm-resolvent limit of the harmonic oscillator Hamiltonian perturbed by a triple of attractive δ -interactions.

1. Introduction

In this note we are going to answer a question that arose naturally in the context of our previous paper [1], that is to say whether the self-adjoint Hamiltonian making rigorous mathematical sense of the formal expression

$$H_\beta = H_0 - \beta |\delta'(x)\rangle\langle\delta'(x)|, H_0 = \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2 \right), \beta \geq 0, \quad (1.1)$$

determined by means of the renormalisation technique used in Theorem 2.1 therein, could also be obtained as the norm resolvent limit of a net of self-adjoint Hamiltonians in which the harmonic oscillator Hamiltonian is perturbed by a suitable configuration of three Dirac distributions in analogy with what was done in [2-4] where the free Hamiltonian was just the negative Laplacian.

Before proceeding further, let us remind the reader that the formal operator (1.1) was rigorously defined in [1] by obtaining the explicit formula for its resolvent (Green’s function), namely:

$$(H_\beta - E)^{-1} = (H_0 - E)^{-1} + \frac{2\beta |\Psi(E)\rangle \langle \Psi(E)|}{1 - 2\beta \left[\frac{1}{2} (H_0 + 1)^{-1}(0,0) - \left(\frac{1}{2} - E \right) (H_0 + 1 - E)^{-1}(0,0) \right]}, \quad (1.2)$$

$$\Psi(x; E) := \sum_{n=0}^{\infty} \frac{(2n+1)^{1/2} \psi_{2n}(0)}{2n + \frac{3}{2} - E} \psi_{2n+1}(x), \quad (1.2a)$$

with a special role played by the quantity in the first summand inside the brackets in the denominator of (1.2), namely

$$\alpha_0 = (H_0 + 1)^{-1}(0,0) = \sum_{n=0}^{\infty} \frac{\psi_{2n}^2(0)}{2n + \frac{3}{2}} = \frac{2\Gamma(3/4)}{\Gamma(1/4)} \cong 0.675978 \quad (1.2b)$$

As was to be expected, the answer to the above question is affirmative. The argument needed to support our claim is essentially based on a suitable modification, based on the findings of [5], of the one used in [4] to show that

$$h_{\{\lambda_\beta(x_0), \mu_\beta(x_0), x_0\}} = -d^2/dx^2 - \mu_\beta(x_0)\delta(x) - \lambda_\beta(x_0)[\delta(x+x_0) + \delta(x-x_0)] \quad (1.3)$$

with $\lambda_\beta(x_0) = \frac{2}{\beta} + \frac{1}{x_0}$, $\mu_\beta(x_0) = \frac{\beta}{x_0^2}$, converges to the negative Laplacian perturbed by the attractive δ' -interaction of strength $-\beta$ in the norm resolvent sense as $x_0 \rightarrow 0_+$.

Before proceeding further, it is worth stressing that, differently from the case of its distributional derivative, namely the δ' , in the case of the Dirac distribution there is no difference between the δ -interaction, that is to say the quadratic form $|\delta\rangle\langle\delta|$ and the so-called δ -potential (see [6,7,8]). Hence, it must be pointed out that the possibility of approximating H_β by means of a suitable triple of attractive Dirac distributions implies the possibility of a potential approximation, as demonstrated by Exner *et al.* in the case of the one-dimensional negative Laplacian perturbed by a single δ' -interaction in [3]. Furthermore, the fact that such Hamiltonians can be approximated by means of those with suitable short-range potentials has a relevance that may go well beyond the mere theoretical interest of mathematical physicists. By citing Cheon's article [2], "this example shows that our procedure to realise the wave function discontinuity is not a mere mathematical abstraction, but something actually realisable in experiments. Recent progress in the quantum device of nanometre scale offers a possible opportunity. We also note that the experiments could be carried out with electromagnetic and other macroscopic waves, since the result obtained here is applicable to larger classes of linear wave equations other than the Schrödinger equation".

With reference to the strategy used in [4], it is certainly worth reminding the reader that, whilst the function $\lambda(x_0)$ had been chosen in such a way to ensure the convergence of the antisymmetric subspace projection operator appearing in the resolvent of the approximant (involving a limit starting from a situation of indetermination of the type 0/0), the choice of $\mu(x_0)$ was fine tuned in order to get exactly the cancellation of the projection onto the symmetric subspace arising from the central delta with the one onto the same subspace due to the twin external deltas.

Before providing the mathematical proof of our statement, some preliminary results will have to be collected.

2. A preparatory lemma

As the behaviour of the symmetric component of the Green functions $(H_0)_s^{-1}(x_0, x_0), (H_0)_s^{-1}(0, x_0)$ and the antisymmetric component $(H_0)_{as}^{-1}(x_0, x_0)$ (adopting the notation used in [5]) in the neighbourhood of $x_0 = 0$ will be crucial in the proof of the main theorem, we are now going to state and prove a lemma regarding some relevant limits of those functions as $x_0 \rightarrow 0_+$. Throughout the following we shall set for any $\beta > 0$:

$$\frac{1}{\lambda_\beta(x_0)} := \frac{2}{\beta} x_0^2 + 2 \sum_{n=0}^{\infty} \frac{\psi_{2n+1}^2(x_0)}{2n + \frac{3}{2}} = \frac{2}{\beta} x_0^2 + 2(H_0)_{as}^{-1}(x_0, x_0) \xrightarrow{x_0 \rightarrow 0_+} 0, \quad (2.1)$$

Lemma 2.1. The following limits hold for any $\beta > 0$:

- i) $\lim_{x_0 \rightarrow 0_+} (H_0)_s^{-1}(x_0, x_0) - (H_0)_{as}^{-1}(x_0, x_0) = \lim_{x_0 \rightarrow 0_+} (H_0)_s^{-1}(x_0, x_0) - \frac{1}{2\lambda_\beta(x_0)} =$
 $= (H_0)^{-1}(0, 0) = \frac{\Gamma(1/4)}{2\Gamma(3/4)} = \frac{1}{\alpha_0} = \frac{1}{(H_0 + 1)^{-1}(0, 0)}$ (2.2)

- ii) $\lim_{x_0 \rightarrow 0_+} \frac{1}{x_0} [(H_0)^{-1}(0, 0) - (H_0)_s^{-1}(x_0, x_0) + (H_0)_{as}^{-1}(x_0, x_0)] =$
 $= \lim_{x_0 \rightarrow 0_+} \frac{1}{x_0} \left[(H_0)^{-1}(0, 0) - (H_0)_s^{-1}(x_0, x_0) + \frac{1}{2\lambda_\beta(x_0)} \right] = 2$ (2.3)

- iii) $\lim_{x_0 \rightarrow 0_+} \frac{(H_0)_{as}^{-1}(x_0, x_0)}{x_0} = \lim_{x_0 \rightarrow 0_+} \frac{1}{2x_0\lambda_\beta(x_0)} =$
 $= \lim_{x_0 \rightarrow 0_+} \frac{(H_0)^{-1}(0, 0) - (H_0)_s^{-1}(x_0, x_0)}{x_0} = 1$ (2.4)

- iv) $\lim_{x_0 \rightarrow 0_+} \frac{(H_0)^{-1}(0, 0) - (H_0)_s^{-1}(x_0, 0)}{x_0} = 1,$
 $\lim_{x_0 \rightarrow 0_+} \frac{(H_0)^{-1}(x_0, 0) - (H_0)_s^{-1}(x_0, x_0)}{x_0} = 0$ (2.5)

Proof. In order to prove i), after using (2.12) and (2.13) in [5], it is straightforward to get:

$$(H_0)_s^{-1}(x_0, x_0) - (H_0)_{as}^{-1}(x_0, x_0) = \frac{1}{\sqrt{\pi}} \int_0^1 \frac{\xi^{-1/2}}{(1 - \xi^2)^{1/2}} e^{-x_0^2 \frac{1+\xi}{1-\xi}} d\xi \quad (2.6)$$

By setting $y = \frac{1+\xi}{1-\xi}$, the right hand side of (2.6) can be rewritten

$$\frac{1}{\sqrt{\pi}} \int_1^{\infty} \frac{e^{-x_0^2 y}}{y^{1/2} (y^2 - 1)^{1/2}} dy \xrightarrow{x_0 \rightarrow 0_+} \frac{1}{\sqrt{\pi}} \int_1^{\infty} \frac{1}{y^{1/2} (y^2 - 1)^{1/2}} dy = \frac{2\Gamma(5/4)}{\Gamma(3/4)}, \quad (2.7)$$

using the dominated convergence theorem. Furthermore,

$$\frac{2\Gamma(5/4)}{\Gamma(3/4)} = \frac{\Gamma(1/4)}{2\Gamma(3/4)} = \frac{1}{\alpha_0} \quad (2.8)$$

Finally, the definition of $\frac{1}{2\lambda_\beta(x_0)}$ easily leads to the other limit in (2.2), fully proving i).

The expression inside the first limit in (2.3) can be written as:

$$\frac{1}{\sqrt{\pi}} \int_1^{\infty} \frac{1 - e^{-x_0^2 y}}{x_0 y^{1/2} (y^2 - 1)^{1/2}} dy = \frac{1}{\sqrt{\pi}} \int_{x_0^2}^{\infty} \frac{1 - e^{-t}}{t^{1/2} (t^2 - x_0^4)^{1/2}} dt, x_0 > 0 \quad (2.9)$$

Taking account that $\frac{1 - e^{-t}}{t^{3/2}} = t^{-1/2} [1 + o(t)]$ as $t \rightarrow 0_+$, the dominated convergence theorem can be exploited again to conclude that (2.9) converges to

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{1 - e^{-t}}{t^{3/2}} dt = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-t}}{t^{1/2}} dt = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = 2 \quad (2.10)$$

as $x_0 \rightarrow 0_+$. The definition of $\frac{1}{2\lambda_\beta(x_0)}$ fully justifies the other limit in (2.3), thus completing the proof of ii).

In order to prove the first limit in iii), we notice that, as a consequence of (2.13) in [5] and the previous transformations, we obtain:

$$\frac{(H_0)_{as}^{-1}(x_0, x_0)}{x_0} = \frac{1}{2\sqrt{\pi}} \int_1^{\infty} \frac{e^{-x_0^2/y} - e^{-x_0^2 y}}{x_0 y^{1/2} (y^2 - 1)^{1/2}} dy \quad (2.11)$$

The right hand side of (2.11) can be transformed into:

$$\frac{1}{2\sqrt{\pi}} \left[\int_1^{\infty} \frac{1 - e^{-x_0^2 y}}{x_0 y^{1/2} (y^2 - 1)^{1/2}} dy - \int_1^{\infty} \frac{1 - e^{-x_0^2/y}}{x_0 y^{1/2} (y^2 - 1)^{1/2}} dy \right] \quad (2.12)$$

Whilst the first integral is exactly the one whose limit has been computed above, the second one is trivially bounded by

$$\frac{1 - e^{-x_0^2}}{x_0} \int_1^{\infty} \frac{1}{y^{1/2} (y^2 - 1)^{1/2}} dy,$$

which obviously vanishes as $x_0 \rightarrow 0_+$. Hence, (2.11) converges to one, thus proving the first limit. The other two limits are an immediate consequence of what has just been obtained.

Finally, in order to show that the first limit in iv) holds, we start by using (2.10) in [5] and the above change of variable to get:

$$\begin{aligned} \frac{(H_0)^{-1}(0,0) - (H_0)_s^{-1}(x_0,0)}{x_0} &= \frac{1}{\sqrt{\pi}} \int_1^\infty \frac{1 - e^{-x_0^2(y+1/y)/4}}{x_0 y^{1/2} (y^2 - 1)^{1/2}} dy = \\ &= \frac{1}{\sqrt{\pi}} \left[\int_1^\infty \frac{1 - e^{-x_0^2/4y}}{x_0 y^{1/2} (y^2 - 1)^{1/2}} dy + \int_1^\infty \frac{e^{-x_0^2/4y} (1 - e^{-x_0^2 y/4})}{x_0 y^{1/2} (y^2 - 1)^{1/2}} dy \right] \end{aligned} \quad (2.13)$$

Then, by using the previous techniques it is quite straightforward to gather that, whilst the first summand vanishes as $x_0 \rightarrow 0_+$, the second one converges to:

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-t/4}}{t^{3/2}} dt = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t/4}}{2t^{1/2}} dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-s^2} ds = 1, \quad (2.14)$$

thus proving our claim. The second limit is an easy consequence of the one just computed and the third limit in iii), which completes the whole proof of the lemma. ■

Before moving forward to the main result of the next section, it might be worth pointing out the following chain of equalities implied by (1.2b), (2.2) and the first resolvent identity:

$$\begin{aligned} \text{tr} \left[\left| (H_0)^{-1}(\cdot, 0) \right\rangle \left\langle (H_0 + 1)^{-1}(0, \cdot) \right| \right] &= \sum_{n=0}^\infty \frac{\psi_{2n}^2(0)}{(2n + \frac{1}{2})(2n + \frac{3}{2})} = \left[(H_0)^{-1}(H_0 + 1)^{-1} \right](0, 0) = \\ &= (H_0)^{-1}(0, 0) - (H_0 + 1)^{-1}(0, 0) = \text{tr} \left[\left| (H_0)^{-1}(\cdot, 0) \right\rangle \left\langle (H_0)^{-1}(0, \cdot) \right| - \left| (H_0 + 1)^{-1}(\cdot, 0) \right\rangle \left\langle (H_0 + 1)^{-1}(0, \cdot) \right| \right] = \\ &= \frac{1}{\alpha_0} - \alpha_0 = \frac{\pi^2 - 2[\Gamma(3/4)]^4}{\sqrt{2}\pi[\Gamma(3/4)]^2} \end{aligned} \quad (2.15)$$

3. The approximation of H_β by means of a suitable triple of attractive Dirac distributions

As a consequence of the uniform estimate (2.15) in [5], for any fixed λ there exists a lower bound E_λ independent of $x_0 > 0$ for all the semibounded operators of the family $H_{\{\lambda, x_0\}}$. Hence, the resolvent of the Hamiltonian of the harmonic oscillator perturbed by a pair of identical attractive delta distributions of strength λ situated at the same distance $x_0 > 0$ from the origin is given by the following direct sum (see also [9-11] for more general configurations of delta distributions perturbing the Hamiltonian of the harmonic oscillator) for any $E < E_\lambda$:

$$\left[H_{\{\lambda, x_0\}} - E \right]^{-1} = \left[(H_0 - E)_s^{-1} + \frac{\left| (H_0 - E)_s^{-1}(\cdot, x_0) \right\rangle \left\langle (H_0 - E)_s^{-1}(x_0, \cdot) \right|}{\frac{1}{2\lambda} - \sum_{n=0}^\infty \frac{\psi_{2n}^2(x_0)}{2n + \frac{1}{2} - E}} \right] \oplus \left[(H_0 - E)_{as}^{-1} + \frac{\left| (H_0 - E)_{as}^{-1}(\cdot, x_0) \right\rangle \left\langle (H_0 - E)_{as}^{-1}(x_0, \cdot) \right|}{\frac{1}{2\lambda} - \sum_{n=0}^\infty \frac{\psi_{2n+1}^2(x_0)}{2n + \frac{3}{2} - E}} \right] \quad (3.1)$$

In view of our approximation the coupling constant will have to be made dependent on x_0 . However, due to the presence of the harmonic confinement, it is clear that the dependence to be used can no longer be the one chosen in the aforementioned papers [2-4] to show that (1.3) does approximate the negative Laplacian perturbed by the attractive δ' -interaction of strength $-\beta$ in the norm resolvent sense as $x_0 \rightarrow 0_+$.

Incidentally, it might be worth noting that this fact seems to have been missed by Cheon and Shigehara in their renowned article [12] on the duality between Bosons and Fermions in one-dimensional Quantum Mechanics. We might come back to this point and some other related issues in a separate paper.

Our choice of $\lambda_\beta(x_0)$ is exactly defined by (2.1), which obviously implies that $\lambda_\beta(x_0) \rightarrow +\infty$ as $x_0 \rightarrow 0_+$. Therefore, the uniform estimate (2.15) of [5] fails to hold given that the coupling is a function of the separation distance. Nevertheless, for any fixed value of the latter parameter, the semibounded self-adjoint Hamiltonian of the harmonic oscillator with the two attractive twin point perturbations symmetrically situated at the distance $x_0 > 0$ from the origin having coupling constant of magnitude (2.1), has its resolvent given by the following direct sum for any E below the ground state energy of $H_{\{\lambda_\beta(x_0), x_0\}}$:

$$\begin{aligned} [H_{\{\lambda_\beta(x_0), x_0\}} - E]^{-1} &= \left[(H_0 - E)_s^{-1} + \frac{|(H_0 - E)_s^{-1}(\cdot, x_0)\rangle\langle (H_0 - E)_s^{-1}(x_0, \cdot)|}{\frac{1}{2\lambda_\beta(x_0)} - \sum_{n=0}^{\infty} \frac{\psi_{2n}^2(x_0)}{2n + \frac{1}{2} - E}} \right] \oplus \\ &\oplus \left[(H_0 - E)_{as}^{-1} + \frac{|(H_0 - E)_{as}^{-1}(\cdot, x_0)\rangle\langle (H_0 - E)_{as}^{-1}(x_0, \cdot)|}{\frac{1}{2\lambda_\beta(x_0)} - \sum_{n=0}^{\infty} \frac{\psi_{2n+1}^2(x_0)}{2n + \frac{3}{2} - E}} \right] \end{aligned} \quad (3.2)$$

We are now going to prove the first of two lemmas that will both be crucial in the proof of our main theorem.

Lemma 3.1. For any $p > 1$, the following limit of the Schatten norm of index p of the difference between the antisymmetric component of the resolvent of H_β and the one of $H_{\{\lambda_\beta(x_0), x_0\}}$, both evaluated at $E = 0$, holds:

$$\left\| [H_\beta]_{as}^{-1} - [H_{\{\lambda_\beta(x_0), x_0\}}]_{as}^{-1} \right\|_p \rightarrow 0 \quad (3.3)$$

as $x_0 \rightarrow 0_+$.

Proof. Due to the cancellation of the antisymmetric component of the resolvent of the harmonic oscillator, the expression inside the norm in (3.3) reduces to the difference between two rank-one operators, so that the norm in (3.3) is bounded from above for $x_0 > 0$ by:

$$\left\| 2\beta |\Psi(0)\rangle\langle \Psi(0)| - \beta \frac{|(H_0)_{as}^{-1}(\cdot, x_0)\rangle\langle (H_0)_{as}^{-1}(x_0, \cdot)|}{x_0^2} \right\|_1, \quad (3.4)$$

having exploited the fact that at the energy $E=0$, the denominator of the second term in (1.2) reduces to one while that of the second term in the second direct summand in (3.2) becomes simply x_0^2 / β , for $\beta > 0$.

It is evident from (3.4) that we must carefully study the behaviour of the sequence

$$\left\{ \frac{\psi_{2n+1}(x_0)}{x_0 \left(2n + \frac{3}{2} \right)} \right\}_{n=0}^{+\infty} \in l_2(\mathbb{N}_0) \quad (3.5)$$

as $x_0 \rightarrow 0_+$. By means of the properties of the eigenfunctions of the harmonic oscillator and their derivatives we get that for any $n \geq 0$, there exists $\xi_{2n+1} \in (0, x_0)$, $x_0 > 0$ such that:

$$\frac{|\psi_{2n+1}(x_0)|}{x_0 \left(2n + \frac{3}{2} \right)} = \frac{|x_0 \psi'_{2n+1}(\xi_{2n+1})|}{x_0 \left(2n + \frac{3}{2} \right)} = \frac{|\sqrt{2(2n+1)} - \xi_{2n+1}| |\psi_{2n}(\xi_{2n+1})|}{\left(2n + \frac{3}{2} \right)} \leq \frac{(\sqrt{2(2n+1)} + x_0) \|\psi_{2n}\|_\infty}{\left(2n + \frac{3}{2} \right)} \quad (3.6)$$

By reminding the reader that $\lim_{n \rightarrow \infty} n^{1/2} \|\psi_n\|_\infty$ exists and is finite (see, e.g., [13] p. 144), we can infer that the right hand side of (3.6) defines a sequence belonging to $l_2(\mathbb{N}_0)$. Furthermore,

$$\left\{ \frac{\psi_{2n+1}(x_0)}{x_0 \left(2n + \frac{3}{2} \right)} \right\}_{n=0}^{+\infty} \rightarrow \left\{ \frac{\sqrt{2(2n+1)} \psi_{2n}(0)}{\left(2n + \frac{3}{2} \right)} \right\}_{n=0}^{+\infty} \in l_2(\mathbb{N}_0) \quad (3.7)$$

as $x_0 \rightarrow 0_+$.

Hence, we need only invoke the dominated convergence theorem to conclude that the convergence of the sequence takes place in norm.

After noticing that the limiting sequence is, apart from the factor $2^{1/2}$, exactly the sequence of the coefficients of the series in (1.2a) at $E=0$, it is not difficult to show that

$$w - \lim_{x_0 \rightarrow 0_+} \beta \frac{|(H_0)_{as}^{-1}(\cdot, x_0) \rangle \langle (H_0)_{as}^{-1}(x_0, \cdot)|}{x_0^2} = 2\beta |\Psi(0) \rangle \langle \Psi(0)| \quad (3.8)$$

and

$$\lim_{x_0 \rightarrow 0_+} \left\| \beta \frac{|(H_0)_{as}^{-1}(\cdot, x_0) \rangle \langle (H_0)_{as}^{-1}(x_0, \cdot)|}{x_0^2} \right\|_1 = 2\beta \|\Psi(0) \rangle \langle \Psi(0)\|_1 \quad (3.9)$$

Hence, by referring to Theorem 2.21 of [11], we get:

$$\left\| [H_\beta]_{as}^{-1} - [H_{\{\lambda_\beta(x_0), x_0\}}]_{as}^{-1} \right\|_p \leq \left\| 2\beta |\Psi(0) \rangle \langle \Psi(0)| - \beta \frac{|(H_0)_{as}^{-1}(\cdot, x_0) \rangle \langle (H_0)_{as}^{-1}(x_0, \cdot)|}{x_0^2} \right\|_1 \rightarrow 0, \quad (3.10)$$

as $x_0 \rightarrow 0_+$, which proves our claim. ■

We must now turn our attention to the other component of the resolvent of the approximating Hamiltonian $H_{\{\lambda_\beta(x_0), x_0\}}$, that is to say the first direct summand on the right hand side of (3.2).

As anticipated in the introduction, our strategy will essentially be identical to the one used in [4] where the free Hamiltonian was the negative one-dimensional Laplacian: given that, for any strictly positive values of both β and x_0 ,

$$\left[H_{\{\lambda_\beta(x_0), \mu(x_0), x_0\}} \right]_s^{-1} = (H_0)_s^{-1} + \frac{|(H_0)_s^{-1}(\cdot, x_0)\rangle\langle(H_0)_s^{-1}(x_0, \cdot)|}{\frac{1}{2\lambda_\beta(x_0)} - (H_0)_s^{-1}(x_0, x_0)} + \frac{\left| \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(\cdot, 0) \right\rangle \left\langle \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(0, \cdot) \right|}{\frac{1}{\mu_\beta(x_0)} - \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(0, 0)},$$

we shall have to choose the coupling constant of the new central delta in such a way to get the cancellation of the symmetric rank-one operators

$$\begin{aligned} & \left\| \frac{\left| \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(\cdot, 0) \right\rangle \left\langle \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(0, \cdot) \right|}{\frac{1}{\mu_\beta(x_0)} - \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(0, 0)} + \frac{|(H_0)_s^{-1}(\cdot, x_0)\rangle\langle(H_0)_s^{-1}(x_0, \cdot)|}{\frac{1}{2\lambda_\beta(x_0)} - (H_0)_s^{-1}(x_0, x_0)} \right\|_1 = \\ & = \left\| \frac{\left| \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(\cdot, 0) \right\rangle \left\langle \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(0, \cdot) \right|}{\frac{1}{\mu_\beta(x_0)} - \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(0, 0)} - \frac{|(H_0)_s^{-1}(\cdot, x_0)\rangle\langle(H_0)_s^{-1}(x_0, \cdot)|}{(H_0)_s^{-1}(x_0, x_0) - \frac{1}{2\lambda_\beta(x_0)}} \right\|_1 \rightarrow 0 \end{aligned} \quad (3.11)$$

as $x_0 \rightarrow 0_+$.

Let us consider first the denominator of the first term inside the norm in (3.11), namely:

$$\begin{aligned} \frac{1}{\mu_\beta(x_0)} - \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(0, 0) &= \frac{1}{\mu_\beta(x_0)} - \left[(H_0)_s^{-1}(0, 0) - \frac{\left[(H_0)_s^{-1}(x_0, 0) \right]^2}{(H_0)_s^{-1}(x_0, x_0) - \frac{1}{2\lambda_\beta(x_0)}} \right] = \\ &= \frac{1}{\mu_\beta(x_0)} - \frac{\left[(H_0)_s^{-1}(x_0, x_0) - \frac{1}{2\lambda_\beta(x_0)} \right] (H_0)_s^{-1}(0, 0) - \left[(H_0)_s^{-1}(x_0, 0) \right]^2}{(H_0)_s^{-1}(x_0, x_0) - \frac{1}{2\lambda_\beta(x_0)}} \end{aligned} \quad (3.12)$$

Taking account of both (2.7) and (2.9) we have:

$$(H_0)_s^{-1}(x_0, x_0) - \frac{1}{2\lambda_\beta(x_0)} = (H_0)^{-1}(0, 0) + 2x_0 + \gamma_1(x_0) \quad (3.13)$$

$$(H_0)_s^{-1}(x_0, 0) = (H_0)^{-1}(0, 0) + x_0 + \gamma_2(x_0), \quad (3.14)$$

where

$$\begin{aligned}\gamma_1(x_0) &= \left[\frac{d^2}{dx_0^2} (H_0)_s^{-1}(x_0, x_0) \right]_{x_0=0} \frac{x_0^2}{2} + o(x_0^2), \\ \gamma_2(x_0) &= \left[\frac{d^2}{dx_0^2} (H_0)_s^{-1}(x_0, 0) \right]_{x_0=0} \frac{x_0^2}{2} + o(x_0^2)\end{aligned}\quad (3.15)$$

Hence, the second term on the right hand side of (3.12) can be written as:

$$\begin{aligned}& \frac{\left[(H_0)^{-1}(0,0) + 2x_0 + \gamma_1(x_0) \right] (H_0)^{-1}(0,0) - \left[(H_0)^{-1}(0,0) + x_0 + \gamma_2(x_0) \right]^2}{(H_0)^{-1}(0,0) + 2x_0 + \gamma_1(x_0)} = \\ &= \frac{-x_0^2 + [\gamma_1(x_0) - 2\gamma_2(x_0)](H_0)^{-1}(0,0) + o(x_0^2)}{(H_0)^{-1}(0,0) + 2x_0 + \gamma_1(x_0)} = \frac{-x_0^2 + \frac{1}{\alpha_0}[\gamma_1(x_0) - 2\gamma_2(x_0)] + o(x_0^2)}{\frac{1}{\alpha_0} + 2x_0 + \gamma_1(x_0)},\end{aligned}\quad (3.16)$$

taking account of (2.2). Finally, we can rewrite the latter expression as:

$$\frac{-\alpha_0 x_0^2 + [\gamma_1(x_0) - 2\gamma_2(x_0)] + \alpha_0 o(x_0^2)}{1 + 2\alpha_0 x_0 + \alpha_0 \gamma_1(x_0)} \quad (3.17)$$

Therefore, it is crucial to understand the behaviour of the difference inside the square brackets in the numerator of the above fraction. By using both the chain rule and the product rule it is not difficult to calculate the second derivatives inside the two expressions in (3.15):

$$\begin{aligned}\left[\frac{d^2}{dx_0^2} (H_0)_s^{-1}(x_0, x_0) \right]_{x_0=0} &= 2 \sum_{n=0}^{\infty} \frac{\psi_{2n}(0) \psi_{2n}''(0)}{2n + \frac{1}{2}}, \\ \left[\frac{d^2}{dx_0^2} (H_0)_s^{-1}(x_0, 0) \right]_{x_0=0} &= \sum_{n=0}^{\infty} \frac{\psi_{2n}(0) \psi_{2n}''(0)}{2n + \frac{1}{2}},\end{aligned}\quad (3.18)$$

which implies that

$$\gamma_1(x_0) - 2\gamma_2(x_0) = o(x_0^2), \text{ as } x_0 \rightarrow 0_+ \quad (3.19)$$

The behaviour of (3.12) as a function of x_0 in the right neighbourhood of 0 is thus expressed by

$$\frac{1}{\mu_\beta(x_0)} - \left[H_{\{\lambda(x_0), x_0\}} \right]^{-1}(0,0) = \frac{1}{\mu_\beta(x_0)} - \frac{-\alpha_0 x_0^2 + o(x_0^2)}{1 + 2\alpha_0 x_0 + \alpha_0 \gamma_1(x_0)} = \frac{1}{\mu_\beta(x_0)} + \alpha_0 x_0^2 + o(x_0^2) \quad (3.20)$$

Let us now turn our attention to the numerator of the first fraction inside the trace class norm, the behaviour of which near 0_+ is determined by the leading term of its expansion, namely:

$$\gamma_3^2(x_0) \frac{|(H_0)_s^{-1}(\cdot, 0)\rangle\langle (H_0)_s^{-1}(0, \cdot)|}{\left[\frac{1}{2\lambda_\beta(x_0)} - (H_0)_s^{-1}(x_0, x_0)\right]^2} \quad (3.21)$$

where

$$\gamma_3(x_0) = \frac{1}{2\lambda_\beta(x_0)} + (H_0)_s^{-1}(x_0, 0) - (H_0)_s^{-1}(x_0, x_0) \quad (3.21a)$$

By noticing that:

$$\gamma_3(x_0) = x_0 \left[\frac{1}{2x_0\lambda_\beta(x_0)} + \frac{(H_0)^{-1}(0, 0) - (H_0)_s^{-1}(x_0, x_0)}{x_0} - \frac{(H_0)^{-1}(0, 0) - (H_0)_s^{-1}(x_0, 0)}{x_0} \right]$$

and taking account of the results of Lemma 2.1, we have

$$\gamma_3(x_0) = x_0[1 + o(x_0)] \text{ as } x_0 \rightarrow 0_+ \quad (3.22)$$

As a consequence, the behaviour of (3.21) in the right neighbourhood of 0 is given by:

$$\gamma_3^2(x_0) \frac{|(H_0)_s^{-1}(\cdot, 0)\rangle\langle (H_0)_s^{-1}(0, \cdot)|}{\left[\frac{1}{2\lambda_\beta(x_0)} - (H_0)_s^{-1}(x_0, x_0)\right]^2} = \alpha_0^2 x_0^2 [1 + o(x_0)]^2 |(H_0)_s^{-1}(\cdot, 0)\rangle\langle (H_0)_s^{-1}(0, \cdot)| \quad (3.23)$$

By combining the above expansion with the one in (3.20) we can resolve the indetermination in the first rank-one operator in the trace class norm in (3.11) provided the coupling constant of the central delta is chosen in a suitable way. In fact, given that

$$\frac{|[H_{\{\lambda_\beta(x_0), x_0\}}]^{-1}(\cdot, 0)\rangle\langle [H_{\{\lambda_\beta(x_0), x_0\}}]^{-1}(0, \cdot)|}{\frac{1}{\mu_\beta(x_0)} - [H_{\{\lambda_\beta(x_0), x_0\}}]^{-1}(0, 0)} = \frac{\alpha_0^2 x_0^2 [1 + o(x_0)]^2 |(H_0)_s^{-1}(\cdot, 0)\rangle\langle (H_0)_s^{-1}(0, \cdot)|}{\frac{1}{\mu_\beta(x_0)} + \alpha_0 x_0^2 + o(x_0^2)} \quad (3.24)$$

by choosing $\mu(x_0) = \beta / x_0^3$ and using de l'Hopital's theorem we obtain:

$$\frac{\alpha_0^2 x_0^2 [1 + o(x_0)]^2 |(H_0)_s^{-1}(\cdot, 0)\rangle\langle (H_0)_s^{-1}(0, \cdot)|}{\alpha_0 x_0^2 + x_0^3 / \beta + o(x_0^2)} \rightarrow \alpha_0 |(H_0)_s^{-1}(\cdot, 0)\rangle\langle (H_0)_s^{-1}(0, \cdot)| \quad (3.25)$$

as $x_0 \rightarrow 0_+$, in the norm topology. The second rank-one operator in the trace class norm also converges to the same limit due to (2.2) and the easy norm convergence of the operator $|(H_0)_s^{-1}(\cdot, x_0)\rangle\langle (H_0)_s^{-1}(x_0, \cdot)|$ to $|(H_0)_s^{-1}(\cdot, 0)\rangle\langle (H_0)_s^{-1}(0, \cdot)|$, confirming what was anticipated in (3.11). Then, for any $p > 1$, we have:

$$\left\| [H_0]_s^{-1} - \left[H_{\{\lambda_\beta(x_0), x_0\}} - \frac{\beta}{x_0^3} \delta(x) \right]_s^{-1} \right\|_p \leq$$

$$\left\| \frac{\left| \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(\cdot, 0) \right\rangle \left\langle \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(0, \cdot) \right|}{\frac{x_0^3}{\beta} - \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]^{-1}(0, 0)} - \frac{\left| (H_0)_s^{-1}(\cdot, x_0) \right\rangle \left\langle (H_0)_s^{-1}(x_0, \cdot) \right|}{(H_0)_s^{-1}(x_0, x_0) - \frac{1}{2\lambda_\beta(x_0)}} \right\|_1 \rightarrow 0, \quad (3.26)$$

which shows that the limit of the symmetric part is simply the symmetric part of the resolvent of the Hamiltonian of the harmonic oscillator evaluated at $E=0$. The result can be summarised in the following lemma.

Lemma 3.2. For any $p > 1$, the following limit of the Schatten norm of index p of the difference between the symmetric component of the resolvent of the Hamiltonian of the harmonic oscillator H_0 and the one of $H_{\{\lambda_\beta(x_0), \mu_\beta(x_0), x_0\}}$, $\mu_\beta(x_0) = \beta/x_0^3$, both evaluated at $E=0$, holds:

$$\left\| \left[H_0 \right]_s^{-1} - \left[H_{\{\lambda(x_0), \mu(x_0), x_0\}} \right]_s^{-1} \right\|_p = \left\| \left[H_0 \right]_s^{-1} - \left[H_{\{\lambda(x_0), x_0\}} - \frac{\beta}{x_0^3} \delta(x) \right]_s^{-1} \right\|_p \rightarrow 0 \quad (3.27)$$

as $x_0 \rightarrow 0_+$.

We are now in the position of stating our main result.

Theorem 3.3. The self-adjoint operator H_β , heuristically defined by (1.1) and rigorously by its resolvent (1.2), is the norm resolvent limit of the net of self-adjoint Hamiltonians

$$H_{\{\lambda_\beta(x_0), \beta/x_0^3, x_0\}} = H_{\{\lambda_\beta(x_0), x_0\}} - \frac{\beta}{x_0^3} \delta(x),$$

$H_{\{\lambda_\beta(x_0), x_0\}}$ being the self-adjoint Hamiltonians rigorously defined by their resolvents (3.2), as $x_0 \rightarrow 0_+$ (x_0 being the distance between the centres of the twin point interactions symmetrically situated with respect to the origin).

Proof. First of all, it might be worth stressing that what is to be shown here is simpler than what was proved in Theorem 2.1 in [1] since the limit of the resolvents is already guaranteed to be the resolvent of a self-adjoint Hamiltonian, namely H_β . Given that, as a consequence of the previous two lemmas, for any $p > 1$, in the limit $x_0 \rightarrow 0_+$:

$$\begin{aligned} \left\| \left[H_\beta \right]^{-1} - \left[H_{\{\lambda_\beta(x_0), \beta/x_0^3, x_0\}} \right]^{-1} \right\|_p &= \left\| \left[H_0 \right]_s^{-1} - \left[H_{\{\lambda_\beta(x_0), \beta/x_0^3, x_0\}} \right]_s^{-1} \right\|_p + \\ &+ \left\| \left[H_\beta \right]_{as}^{-1} - \left[H_{\{\lambda_\beta(x_0), x_0\}} \right]_{as}^{-1} \right\|_p \rightarrow 0, \end{aligned} \quad (3.28)$$

we also have:

$$\left\| \left[H_\beta \right]^{-1} \right\|_p - \left\| \left[H_{\{\lambda_\beta(x_0), \beta/x_0^3, x_0\}} \right]^{-1} \right\|_p \rightarrow 0 \quad (3.29)$$

as $x_0 \rightarrow 0_+$. The latter immediately implies that $\forall \varepsilon > 0$, there exists a particular value $X_0(\varepsilon) > 0$ of the separation distance such that $\forall x_0$ inside the interval $(0, X_0(\varepsilon))$ the following bounds hold:

$$\left\| [H_\beta]^{-1} \right\|_p - \varepsilon < \left\| [H_{\{\lambda_\beta(x_0), \beta/x_0^3, x_0\}}]^{-1} \right\|_p < \left\| [H_\beta]^{-1} \right\|_p + \varepsilon \quad (3.30)$$

By exploiting (2.3a) in [1], it is possible to estimate the Schatten norm with index p of $[H_\beta]^{-1}$ as follows:

$$\begin{aligned} \left\| [H_\beta]^{-1} \right\|_p &\leq \left\| [H_0]^{-1} \right\|_p + 2\beta \left\| \Psi(0) \right\rangle \left\langle \Psi(0) \right\|_1 = \\ &= \sum_{n=0}^{\infty} \frac{1}{\left(n + \frac{1}{2}\right)^p} + 2\beta \left\| \Psi(0) \right\|_2^2 \leq \sum_{n=0}^{\infty} \frac{1}{\left(n + \frac{1}{2}\right)^p} + 2\beta (H_0 + 1)^{-1}(0,0) = \sum_{n=0}^{\infty} \frac{1}{\left(n + \frac{1}{2}\right)^p} + 2\alpha_0 \beta \end{aligned} \quad (3.31)$$

Let us now consider $E_\varepsilon = i\gamma_\varepsilon$, with $0 < \gamma_\varepsilon < \left[\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{-p} + 2\alpha_0 \beta + \varepsilon \right]^{-1}$. Then, $\forall x_0$ inside the interval $(0, X_0(\varepsilon))$ the following bounds hold

$$\gamma_\varepsilon \left\| [H_{\{\lambda_\beta(x_0), \beta/x_0^3, x_0\}}]^{-1} \right\|_p < 1, \gamma_\varepsilon \left\| [H_\beta]^{-1} \right\|_p < 1 \quad (3.32)$$

The latter bounds immediately imply the existence of the bounded operators

$$\left[I + i\gamma_\varepsilon [H_{\{\lambda_\beta(x_0), \beta/x_0^3, x_0\}}]^{-1} \right]^{-1}$$

and

$$\left[I + i\gamma_\varepsilon [H_\beta]^{-1} \right]^{-1}.$$

Hence, as $x_0 \rightarrow 0_+$, the first resolvent identity implies that:

$$\begin{aligned} &\left[H_{\{\lambda_\beta(x_0), \beta/x_0^3, x_0\}} - i\gamma_\varepsilon \right]^{-1} = \\ &= \left[H_{\{\lambda_\beta(x_0), \beta/x_0^3, x_0\}} \right]^{-1} \left[I + i\gamma_\varepsilon [H_{\{\lambda_\beta(x_0), \beta/x_0^3, x_0\}}]^{-1} \right]^{-1} \rightarrow [H_\beta]^{-1} \left[I + i\gamma_\varepsilon [H_\beta]^{-1} \right]^{-1} = \\ &= [H_\beta - i\gamma_\varepsilon]^{-1} \end{aligned} \quad (3.33)$$

Having established the norm convergence of the resolvents at the point $E_\varepsilon = i\gamma_\varepsilon$, we need only invoke Theorem VIII. 19 in [14] to complete our proof. ■

Before ending this section we wish to provide the reader with a graph of $E_1(\lambda_{\beta=2}(x_0), x_0)$, the lowest odd eigenvalue of $H_{\{\lambda_2(x_0), x_0\}} - \frac{2}{x_0^3} \delta(x)$ clearly converging, as the distance between the two external deltas shrinks to zero, to $E_1(\beta = 2) \cong 0.393575$, the lowest odd eigenvalue of the Hamiltonian $H_{\beta=2}$, which is

actually the ground state energy of the Hamiltonian given that the value $\beta = 2 > \frac{1}{\alpha_0}$ is to the right of the point where the level crossing pointed out in [1] occurs.

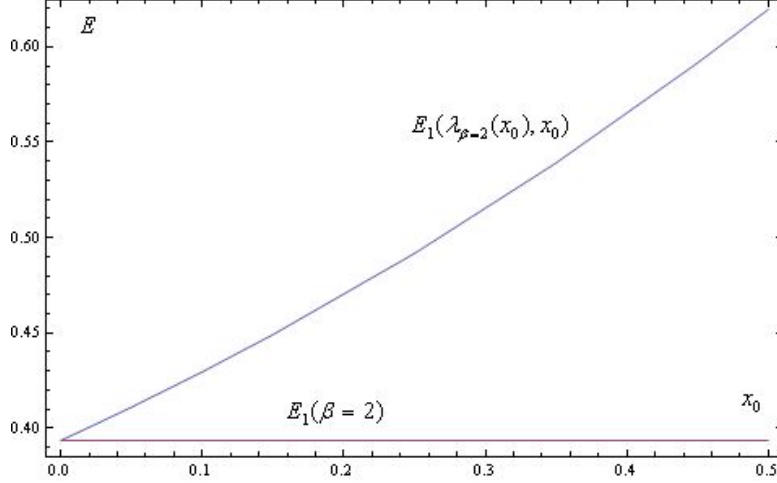


Fig. 1. The lowest odd eigenvalue $E_1(\lambda_{\beta=2}(x_0), x_0)$ approaching $E_1(\beta = 2) \cong 0.393575$ as $x_0 \rightarrow 0_+$.

4. Final remarks

In this note we have provided an alternative procedure to define the self-adjoint Hamiltonian

$$H_\beta = H_0 - \beta |\delta'(x)\rangle\langle\delta'(x)|, H_0 = \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2 \right), \beta \geq 0$$

introduced and investigated in [1] using the renormalisation of the coupling constant. Here instead, mimicking what was done in [3,4] in the case of

$$h_\beta = -d^2 / dx^2 - \beta |\delta'(x)\rangle\langle\delta'(x)|,$$

we have been able to show that H_β can be approximated in the norm resolvent sense by the self-adjoint Hamiltonian of the harmonic oscillator perturbed by a suitable triple of attractive delta distributions. Since each delta distribution can in turn be approximated by a suitable sequence of short-range potentials with shrinking support, our result paves the way to a potential approximation, even in the presence of the harmonic confinement, of the highly singular δ' -interaction that can be regarded, as has been stressed in [1,15], as a momentum dependent interaction.

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