Work in preparation:

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on RELATIVE PARTITION FUNCTION FOR COULOMB + POINT INTER.

$$H_0 = -\Delta + \frac{\gamma}{|x|}$$

in $L^2(\mathbb{R}^3)$, $\gamma \in \mathbb{R}$

Well def., s.a., lower bounded

$$R(\lambda; H_0)(x,y) = -\frac{\Gamma\left(1 - \frac{\gamma}{2\sqrt{-\lambda}}\right)}{4\pi |x-y|}$$

$$K\left(-\frac{\gamma}{2\sqrt{-\lambda}}; \sqrt{-\lambda}\, x_+; \sqrt{-\lambda}\, x_-\right), \text{ with}$$

$$K(a; b; c) := W_{a,\frac{1}{2}}(b) M'_{a,\frac{1}{2}}(c) - W'_{a,\frac{1}{2}}(b) M_{a,\frac{1}{2}}(c)$$

$\lambda \in \rho(H_0)$, $\operatorname{Re}\sqrt{-\lambda} > 0$, $x, y \in \mathbb{R}^3$

$$x_\pm := |x| + |y| \pm |x-y|$$

M, W: Whittaker functions

$\Gamma(s) := \int_0^{+\infty} x^{s-1} e^{-x}\,dx$, $\mathrm{Re}\, s > 0 \ldots$

Prop.1 i) $\sigma_{ess}(H_0) = \sigma_{ac}(H_0) = [0, +\infty)$

ii) If $\gamma \geq 0$: $\sigma_{pp}(H_0) = \emptyset$

If $\gamma < 0$: $\sigma_{pp}(H_0) = -\frac{\gamma^2}{4(n+1)^2}$,

$n \in \mathbb{N}_0$

$$H_\alpha = \text{"} -\Delta + \frac{\gamma}{|x|} + \alpha\,\delta_0(x) \text{"}$$

$-\infty < \alpha \leq +\infty$, $\gamma \in \mathbb{R}$.

rigorously given as lower bounded, s.a. operator by "resolvent kernel"

$$R(\lambda; H_\alpha)(x,y) = R(\lambda; H_0)(x,y)$$

$$+ \frac{4\pi}{4\pi\alpha - \gamma F_\gamma\left(\frac{\gamma}{2\sqrt{-\lambda}}\right)}\, g(\lambda; x)\, g(\lambda; y),$$

$x, y \in \mathbb{R}^3$

$\lambda \in \rho(H_\alpha) \cap \rho(H_0)$, $\mathrm{Re}\sqrt{-\lambda} > 0$

$$g(\lambda; x) := \frac{\Gamma\left(1 + \frac{\gamma}{2\sqrt{-\lambda}}\right)}{4\pi|x|} W_{-\frac{\gamma}{2\sqrt{-\lambda}}, \frac{1}{2}}(2\sqrt{-\lambda}|x|),$$

$x \neq 0$

$$F_\gamma(z) := \begin{cases} \psi(1+z) - \log z - \frac{1}{2z} - \psi(1) - \psi(2), & \gamma > 0 \\ \psi(1+z) - \log(-z) - \frac{1}{2z} - \psi(1) - \psi(2), & \gamma < 0, \end{cases}$$

$z \in \mathbb{C}$

$$\psi(z) := \frac{d}{dz} \log \Gamma(z), \quad z \in \mathbb{C}$$

"digamma function"

<u>Prop. 2</u> i) $\sigma_{ess}(H_\alpha) = \sigma_{ac}(H_\alpha) = [0, +\infty)$

ii) Eigenvalues $E < 0$ of H_α "for $l = 0$" solve

$$4\pi\alpha = \gamma F_\gamma\left(\frac{\gamma}{2\sqrt{-E}}\right)$$

iii) For $\gamma \geq 0$, $\alpha > -\gamma\,[\psi(1)+\psi(2)]/4\pi$

$$\sigma_{pp}(H_\alpha) = \emptyset$$

iv) For $\gamma \geq 0$, $\alpha < -\gamma\,[\psi(1)+\psi(2)]/4\pi$

$$|\sigma_{pp}(H_\alpha)| = 1$$

(eigenvalue given by above eqt.)

v) $\gamma < 0$: ∞-many solutions of above eqt., simple eigenvalues corresp. to $l = 0$

For $l \geq 1$: Eigenvalues

$$-\frac{\gamma^2}{4m^2}, \quad m \geq 2, \; m \in \mathbb{N}$$

Consider now the case $\gamma \geq 0$

Prop. 3 The "relative trace"

$r(\lambda; H_\alpha, H_0)$ is well def. for $\lambda \in \rho(H_0) \cap \rho(H_\alpha)$,

$$r(\lambda; H_\alpha, H_0) = \mathrm{Tr}\,[R(\lambda; H_\alpha) - R(\lambda; H_0)]$$

One has:

$$r(\lambda; H_\alpha, H_0) =$$

$$= - \frac{z\left[2\psi'(1+z)z^2 - 2z + 1\right]}{A(\gamma, \alpha, z)},$$

with $A(\gamma, \alpha, z) := \gamma(4\pi\alpha - \gamma)\left(\psi(1+z) - \log z - \frac{1}{2z} - \psi(1) - \psi(2)\right)$,

with $\mathrm{Re}\sqrt{-\lambda} > 0$, $z := \frac{\gamma}{2\sqrt{-\lambda}}$.

Proof: Use resolvent formula, integration, Γ-functional equation...

Rem: Using asympt. exp. of digamma-function ψ one gets:

$$r(\lambda; H_\alpha, H_0) = \sum_{k=0}^{\infty} b_k (-\lambda)^k, \quad b_k \in \mathbb{R}, \quad |\lambda| \ll 1$$

e.g. $b_0 = \left[3\gamma(\gamma - 2C\gamma + 4\pi\alpha)\right]^{-1}$

$$b_1 = \frac{(17 - 24C)\gamma + 48\pi\alpha}{45\gamma^3(2C\gamma - \gamma - 4\pi\alpha)^2}$$

$$\sum_{j=2,k=0}^{\infty} a_{j,k} (-\lambda)^{-\frac{1}{2}} (\log(-\lambda))^k \quad , \quad |\lambda| \gg 1$$

$a_{j,k} \in \mathbb{R}$

$a_{2,0} = -\frac{1}{2}, \quad a_{2,k>0} = 0,$

$a_{3,0} = \frac{1}{2} \left[4\pi\alpha + (2-C)\gamma + \gamma(\log\gamma - \log 2) \right.$

$a_{3,1} = -\frac{\gamma}{4}$

$a_{3,k>1} = 0.$

One gets corresponding formulae for "relative spectral measure $e(v; H_\alpha, H_0)$, $v \in (0, +\infty)$, s.t., e.g.

$$\mathrm{Tr}\left(e^{-tH_\alpha} - e^{-tH_0}\right) =$$

$$= \int_0^\infty e^{-v^2 t} e(v; H_\alpha, H_0)\, dv$$

Namely:

$$e(v; H_\alpha, H_0) = \frac{v}{i\pi} \lim_{\varepsilon \downarrow 0+} \left[r(v^2 e^{2\pi i - i\varepsilon}; H_\alpha, H_0) - r(v^2 e^{i\varepsilon}; H_\alpha, H_0) \right],$$

with "explicit expansions" for $v \downarrow 0$, $v \uparrow +\infty$.