A Remark on the Hamiltonian for Three Identical Bosons with Zero-Range Interactions

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Trails in Quantum Mechanics and Surroundings

Outline

Main References

- [FM] L. Faddeev, R.A. Minlos, *Soviet Phys. Dokl.*, 6 (1962).
- [MeM] A.M. Melnikov, R.A. Minlos, *Adv. Soviet Math.*, 5 (1991).
- [CDFMT] M. Correggi, G. Dell'Antonio, D. Finco, A. Michelangeli, A. Teta , *Rev. Math. Phys.*, 24 (2012).
- [CFT] M. Correggi, D. Finco, A. Teta, *Europhysics Letters*, (2015).

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2. Explicit Construction
3. Fermionic Case
4. Bosonic Case

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Zero-Range Interaction

System of three particles interacting via zero-range two-body interactions.
Formally

$$\mathcal{H} = -\sum_{i=1}^{3} \frac{1}{2m_i}\Delta_{\mathbf{x}_i} + \sum_{i<j} \gamma_{ij}\delta(\mathbf{x}_i - \mathbf{x}_j)$$

where $\mathbf{x}_i \in \mathbb{R}^3$, $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathbb{R}^9$ and $\gamma_{ij} \in \mathbb{R}$.

Motivations

- Nuclear Physics
- Ultra-Cold Quantum Gases

We look for a rigorous definition: self-adjoint operator (crucial to define the dynamics), possibly bounded from below (in order to ensure the stability of the system).

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Rigorous Definition

Note that

$$\mathcal{H}\psi = \mathcal{H}_0\psi \text{ if } \psi \text{ vanishes on each hyperplane } \{\mathbf{x}_i = \mathbf{x}_j\}$$

So we consider

$$\widetilde{\mathcal{H}_0} = -\frac{1}{2m_i}\sum_{i=1}^{3}\Delta_{\mathbf{x}_i} \qquad \mathcal{D}(\widetilde{\mathcal{H}_0}) = C_0^\infty\Big(\mathbb{R}^9 \setminus \bigcup_{i<j}\{\mathbf{x}_i = \mathbf{x}_j\}\Big)$$

symmetric but not self-adjoint.

$\Longrightarrow$ by definition any self-adjoint extension of $\widetilde{\mathcal{H}_0}$, different from $\mathcal{H}_0$, is a Hamiltonian for a system of three particles in $\mathbb{R}^3$ with zero-range interactions.

The problem is to give an explicit construction.

$\Longrightarrow$ We proceed by analogy with the well known two-body case.

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STM Extensions

STM extensions $\mathcal{H}_a$ of $\widetilde{\mathcal{H}}_0$:

$$\mathcal{H}_a\psi = \mathcal{H}_0\psi \quad \text{if } \mathbf{x}_i \neq \mathbf{x}_j$$

and $\psi \in \mathcal{D}(\mathcal{H}_a)$ satisfies the boundary conditions

$$\psi \underset{|\mathbf{x}_i - \mathbf{x}_j| \to 0}{\simeq} \left(\frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} - \frac{1}{a} \right) Q_{ij}(\mathbf{r}_{ij}, \mathbf{x}_k) + o(1), \qquad \mathbf{r}_{ij} = \frac{m_i \mathbf{x}_i + m_j \mathbf{x}_j}{m_i + m_j}.$$

In general $\mathcal{H}_a$ is neither self-adjoint nor bounded from below and its s.a. extensions are in general unbounded from below (*Thomas effect* [FM], [MeM]).

We show s.a. and boundness from below for two systems using the quadratic form ($a = \infty$):

$$\mathcal{F}[\psi] = (\psi, \mathcal{H}\psi) \qquad \text{for any } \psi \in \mathcal{D}(\mathcal{H})$$

$\Longrightarrow$ If $\mathcal{F}$ is closed and bounded from below then $\mathcal{H}$ is self-adjoint and bounded from below.

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The Potential Produced by Q_{ij}

Let us introduce the "potential" produced by the "charges" Q_{ij}:

$$(\mathcal{G}Q)(\mathbf{X}) = \sum_{i<j} \frac{1}{(2\pi)^5 \mu_{ij}} \int d\mathbf{P} e^{i\mathbf{X}\cdot\mathbf{P}} \frac{\hat{Q}_{ij}(\mathbf{p}_i + \mathbf{p}_j, \mathbf{p}_k)}{h_0(\mathbf{P})}$$

where $h_0(\mathbf{P}) = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{p}_3^2}{2m_3}$ and $\mu_{ij} = \frac{m_i m_j}{m_i + m_j}$.

Properties

- $\mathcal{H}_0(\mathcal{G}Q)(\mathbf{X}) = 2\pi \sum_{ij} \frac{1}{\mu_{ij}} Q_{ij}(\mathbf{r}_{ij}, \mathbf{x}_k)\delta(\mathbf{x}_i - \mathbf{x}_j)$
- $(\mathcal{G}Q)(\mathbf{X}) \simeq \frac{Q_{ij}(\mathbf{r}_{ij}, \mathbf{x}_k)}{|\mathbf{x}_i - \mathbf{x}_j|} - (\Omega Q)_{ij}(\mathbf{r}_{ij}, \mathbf{x}_k) + o(1)$ if $|\mathbf{x}_i - \mathbf{x}_j| \to 0$

Writing

$$\psi = w + \mathcal{G}Q \qquad w \text{ smooth}$$

the boundary conditions become

$$\lim_{|\mathbf{x}_i - \mathbf{x}_j| \to 0} w(\mathbf{X}) = (\Omega Q)_{ij}(\mathbf{r}_{ij}, \mathbf{x}_k).$$

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Quadratic Form

Defining $\mathcal{D}_\varepsilon = \{\mathbf{X} \in \mathbb{R}^9 : |\mathbf{x}_i - \mathbf{x}_j| > \varepsilon\}$ we have

$$
\begin{aligned}
(\psi, \mathcal{H}\psi) &= \lim_{\varepsilon \to 0} \int_{\mathcal{D}_\varepsilon} d\mathbf{X}\, \overline{\psi} \mathcal{H}_0 \psi \\
&= (w, \mathcal{H}_0 w) + \lim_{\varepsilon \to 0} \int_{\mathcal{D}_\varepsilon} d\mathbf{X}\, \overline{\mathcal{G}Q}\, \mathcal{H}_0 w \\
&= (w, \mathcal{H}_0 w) + \sum_{i<j} \frac{2\pi}{\mu_{ij}} \int d\mathbf{p} d\mathbf{k} |\xi_{ij}(\mathbf{k}, \mathbf{p})|^2 \sqrt{\frac{\mu_{ij} M}{m_k(m_i + m_j)} k^2 + \frac{\mu_{ij}}{M} p^2} \\
&\quad - \sum_{ijk} \frac{1}{2\pi \mu_{ij} \mu_{jk}} 2\Re \left(\int d\mathbf{p} d\mathbf{k}_{ij} d\mathbf{k}_{jk} \frac{\overline{\xi_{ij}(\mathbf{k}_{jk}, \mathbf{p})} \xi_{jk}(\mathbf{k}_{ij}, \mathbf{p})}{\frac{k_{ij}^2}{2\mu_{ij}} + \frac{k_{jk}^2}{2\mu_{jk}} + \frac{\mathbf{k}_{ij} \cdot \mathbf{k}_{jk}}{m_j} + \frac{p^2}{2M}} \right)
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where $\xi_{ij}(\mathbf{k}, \mathbf{p}) = \hat{Q}_{ij}(\frac{m_i + m_j}{M} \mathbf{p} - \mathbf{k}, \frac{m_k}{M} \mathbf{p} + \mathbf{k})$ with $M = \sum_i m_i$.

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2 Fermions + 1

Let us consider two identical fermions (e.g. 1 and 3) of mass m interacting with a different particle (e.g. 2) of mass m_0. Let $\alpha = \frac{m}{m_0}$ the mass ratio.

- Hilbert space: $L^2_{\mathrm{asym}}(\mathbb{R}^9)$
- $\xi_{13} = 0$ and $\xi_{12} = -\xi_{23} = \xi$ because of antisymmetry

Energy form

$$\mathcal{D}[\mathcal{F}] = \{\psi \in L^2_{\mathrm{asym}}(\mathbb{R}^9) : \psi = w + \mathscr{G}\xi, w \in H^1_{\mathrm{asym}}(\mathbb{R}^9), \xi \in H^{1/2}(\mathbb{R}^6)\}$$

$$\mathcal{F}[\psi] = \mathcal{F}_0[w] + \frac{2(1+\alpha)}{m\pi}\Phi[\xi]$$

where $\mathcal{F}_0[w] = (w, \mathcal{H}_0 w)$ and
$\Phi[\xi] = \int d\mathbf{p}\Phi_{\mathbf{p}}[\xi] = \int d\mathbf{p}({\Phi_{\mathbf{p}}}^{\mathrm{diag}}[\xi] + {\Phi_{\mathbf{p}}}^{\mathrm{off}}[\xi])$ is a form acting on the "charge" ξ.

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Stability

The following theorem has been proved [CDFMT]:

Theorem

The energy form is positive and closed if and only if $\alpha \leq \alpha_c$ where α_c is the solution of the equation

$$\Lambda(\alpha) = \frac{2}{\pi}\left(\frac{1+\alpha}{\alpha}\right)^2 \left[\frac{\alpha}{\sqrt{1+2\alpha}} - \arcsin\left(\frac{\alpha}{1+\alpha}\right)\right] = 1$$

This implies that $\mathcal{H}$ is self-adjoint and bounded from below.

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3 Bosons

Let us now consider a system composed by three identical bosons of mass 1.

- Hilbert space: $L^2_{\text{sym}}(\mathbb{R}^9)$
- $\xi_{ij} = \xi$ for all i, j.

Energy form:

$$\mathcal{D}[\mathcal{F}] = \{\psi \in L^2_{\text{sym}}(\mathbb{R}^9) : \psi = w + \mathscr{G}\xi, w \in H^1_{\text{sym}}(\mathbb{R}^9), \xi \in H^{1/2}(\mathbb{R}^6)\}$$

$$\mathcal{F}[\psi] = (w, \mathcal{H}_0 w) + \frac{6}{\pi}\Phi[\xi]$$

where $\Phi[\xi] = \int d\mathbf{p}\Phi_{\mathbf{p}}[\xi] = \int d\mathbf{p}(\Phi_{\mathbf{p}}{}^{\text{diag}}[\xi] + \Phi_{\mathbf{p}}{}^{\text{off}}[\xi])$ with

- $\Phi_{\mathbf{p}}{}^{\text{diag}}[\xi] = 2\pi^2 \int d\mathbf{k} |\xi(\mathbf{k},\mathbf{p})|^2 \sqrt{\frac{3}{4}k^2 + \frac{p^2}{6}}$
- $\Phi_{\mathbf{p}}{}^{\text{off}}[\xi] = -2\int d\mathbf{k}_1\, d\mathbf{k}_2 \frac{\overline{\xi(\mathbf{k}_1,\mathbf{p})}\xi(\mathbf{k}_2,\mathbf{p})}{k_1^2 + k_2^2 + \mathbf{k}_1\cdot\mathbf{k}_2 + \frac{p^2}{6}}$

Such a form is well known to be unbounded.

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$$\mathcal{D}[\mathcal{F}] = \{\psi \in L^2_{\text{sym}}(\mathbb{R}^9) : \psi = w + \mathscr{G}\xi, w \in H^1_{\text{sym}}(\mathbb{R}^9), \xi \in H^{1/2}(\mathbb{R}^6)\}$$

$$\mathcal{F}[\psi] = (w, \mathcal{H}_0 w) + \frac{6}{\pi}\Phi[\xi]$$

where $\Phi[\xi] = \int d\mathbf{p}\Phi_{\mathbf{p}}[\xi] = \int d\mathbf{p}(\Phi_{\mathbf{p}}{}^{\text{diag}}[\xi] + \Phi_{\mathbf{p}}{}^{\text{off}}[\xi])$ with

- $\Phi_{\mathbf{p}}{}^{\text{diag}}[\xi] = 2\pi^2 \int d\mathbf{k}|\xi(\mathbf{k},\mathbf{p})|^2 \sqrt{\frac{3}{4}k^2 + \frac{p^2}{6}}$
- $\Phi_{\mathbf{p}}{}^{\text{off}}[\xi] = -2\int d\mathbf{k}_1\, d\mathbf{k}_2 \frac{\overline{\xi(\mathbf{k}_1,\mathbf{p})}\xi(\mathbf{k}_2,\mathbf{p})}{k_1^2 + k_2^2 + \mathbf{k}_1\cdot\mathbf{k}_2 + \frac{p^2}{6}}$

Such a form is well known to be unbounded.

3 Bosons

Let us now consider a system composed by three identical bosons of mass 1.

- Hilbert space: $L^2_{\text{sym}}(\mathbb{R}^9)$
- $\xi_{ij} = \xi$ for all i, j.

Energy form:

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Such a form is well known to be unbounded.

Main Result

Theorem

$\Phi[\xi]$ is positive for

$$\xi(\mathbf{k},\mathbf{p}) = \sum_{\substack{m,l \\ l>0}} \xi_{lm}(k,\mathbf{p}) Y_{lm}(\theta,\phi).$$

Note:

- $\Phi_{\mathbf{p}}{}^{diag}[\xi]$ positive
- $\Phi_{\mathbf{p}}{}^{off}[\xi]$ not definite in sign

$\Rightarrow$ Crucial point: $\Phi_{\mathbf{p}}{}^{off}[\xi] \geq -\Gamma \Phi_{\mathbf{p}}{}^{diag}[\xi]$ with $\Gamma < 1$.

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Sketch of the Proof $\Phi_{\mathbf{p}}^{\text{off}}[\xi] \geq -\Gamma \Phi_{\mathbf{p}}^{\text{diag}}[\xi]$ [I]:

$$\Phi_{\mathbf{p}}^{\text{off}}[\xi] = -2 \int d\mathbf{k}_1 \, d\mathbf{k}_2 \frac{\overline{\xi(\mathbf{k}_1, \mathbf{p})}\xi(\mathbf{k}_2, \mathbf{p})}{k_1^2 + k_2^2 + \mathbf{k}_1 \cdot \mathbf{k}_2 + \frac{p^2}{6}}$$

[exp. in spherical harmonics]

$$= -4\pi \sum_{lm, l \neq 0} \int_0^\infty dk_1 \int_0^\infty dk_2 \, k_1^2 \overline{\xi_{lm}(k_1, \mathbf{p})} k_2^2 \xi_{lm}(k_2, \mathbf{p}) \times \int_{-1}^1 dy \frac{P_l(y)}{k_1^2 + k_2^2 + k_1 k_2 y + \frac{p^2}{6}}$$

[terms with l odd are positive, terms with l even are decreasing in p]

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$$\left[\text{diagonalization using: } \xi^{\sharp}_{lm}(q,\mathbf{p}) = \frac{1}{2\pi}\int dx e^{-ikx} e^{2x} \xi_{lm}(e^x,\mathbf{p})\right]$$

$$= - \sum_{\substack{lm, l\neq 0 \\ l \text{ even}}} \int dq S_l(q) |\xi^{\sharp}_{lm}(q,\mathbf{p})|^2$$

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Thank you for the attention!

Appendix $S_2(q) \leq \Delta$

To obtain Δ :

$$S_2(q) = \pi^2 \int_{-1}^{1} dy (3y^2 - 1) \frac{\cosh(q \arcsin \frac{y}{2})}{\cos(\arcsin \frac{y}{2}) \cosh(q \frac{\pi}{2})}$$

So

$$S_2(0) = \pi^2 \int_{-1}^{1} dy (3y^2 - 1) \frac{1}{\cos(\arcsin(\frac{y}{2}))} = 2\pi^2 \left(\frac{5}{3}\pi - 3\sqrt{3} \right)$$

Thus it's enough to show $S_2(q) - S_2(0) \leq 4\pi^2 \frac{b}{9}$, $q \geq 0$ (even function).

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Introduce

$$g(q) = a + \left(\frac{10}{9}b - a\right)\cosh\left(q\frac{\pi}{2}\right) - b\cosh\left(q\frac{\pi}{6}\right)$$

then $I \leq -\frac{g(q)}{\cosh(q\frac{\pi}{2})} + \frac{b}{9}$.

- $g(0) = \frac{b}{9} > 0$
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