Schrödinger equation with nonlinear defect as effective model

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Joint work with C.Cacciapuoti, D.Noja, A.Teta
(first part of the talk refers to *Lett.Math.Phys.* (**104**) (2014))

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Preliminaries
One dimensional problem
Three dimensional problem

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- Widely used in physics for diffraction of electrons from a thin layer, analysis of nonlinear resonant tunneling, models of soliton bifurcation and so on

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- The NLS is an ubiquitous equation describing several nonlinear phenomena
- There are simplified models where the nonlinearity is concentrated at point
- Widely used in physics for diffraction of electrons from a thin layer, analysis of nonlinear resonant tunneling, models of soliton bifurcation and so on
- Rigorous analysis concerning existence of dynamics, blow up, asymptotic stability [Adami, Dell'Antonio, Figari, Noja, Ortoleva, Teta,...]

At a formal level, we are considering the equation

$$i\frac{d}{dt}\psi(t) = -\Delta\psi(t) + \gamma\,\delta_0|\psi(t)|^{2\mu}\psi(t)$$

Here we discuss the approximation problem: we want to discuss how this equation appears as limit of a rescaled NLS.

The one dimensional case and the three dimensional case are quite different.

1d linear case

Self-adjoint operator H_α

$$H_\alpha = -\frac{d^2}{dx^2} + \alpha\delta_0 \qquad \alpha \in \mathbb{R}$$

$$\mathcal{D}(H_\alpha) \;=\; \left\{\psi \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R}),\; \psi'(0+) - \psi'(0-) = \alpha\psi(0)\right\}$$

$$H_\alpha\psi \;= -\psi'' \qquad \forall x \neq 0$$

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Representation for the unitary group generated by H_α

$$\begin{cases} i\dfrac{d}{dt}\psi(t) = H_\alpha\psi(t) \\ \psi(0) = \psi_0 \end{cases} \qquad \psi_0 \in H^1$$

$$\psi(t,x) = (U(t) * \psi_0)(x) - i\int_0^t \alpha U(t-s,x)\psi(s,0)ds$$

$$U(t,x) = \frac{1}{\sqrt{4\pi i t}} e^{i\frac{x^2}{4t}}$$

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$$\mathcal{E}(\psi(t)) = \int dx|\psi'(t,x)|^2 + \frac{\gamma}{\mu+1}|\psi(t,0)|^{2\mu+2}$$

Theorem

This equation has a global solution for $\psi_0 \in H^1(\mathbb{R})$ if $\gamma > 0$ and $\forall \mu > 0$ or if $\gamma < 0$ and $0 < \mu < 1$. Moreover energy is conserved along the solutions.

Approximating problem

We know that in one dimension

$$-\frac{d}{dx^2}+\frac{1}{\varepsilon}V\left(\frac{x}{\varepsilon}\right)\longrightarrow -\frac{d}{dx^2}+\alpha\delta_0 \quad \alpha=\int V(x)$$

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$$\psi^\varepsilon(t,x)=U(t)\psi_0(x)-i\int_0^t ds\int dy\, U(t-s,x-y)\,\frac{1}{\epsilon}V\left(\frac{y}{\epsilon}\right)\,|\psi^\varepsilon(s,y)|^{2\mu}\psi^\varepsilon(s,y)$$

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Also for the approximating problem energy is conserved

$$\mathcal{E}^\varepsilon(\psi^\varepsilon(t))=\int dx|\psi^{\varepsilon\prime}(t,x)|^2+\frac{1}{\mu+1}\int_{\mathbb{R}} dx\,\frac{1}{\varepsilon}V\left(\frac{x}{\varepsilon}\right)\,|\psi^\varepsilon(t,x)|^{2\mu+2}$$

Main theorem in one dimension

The main result is the following

Theorem

Take $V \in L^1(\mathbb{R}, \langle x \rangle \, dx) \cap L^\infty(\mathbb{R})$ *and* $\gamma = \int V dx$. *Let* $V \geq 0$ *or* $\mu \in (0,1)$ *then for any* $T > 0$ *we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|\psi^\varepsilon(t) - \psi(t)\|_{H^1} = 0$$

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- under the above hypothesis both the approximating problem and the limit one have global solutions
- notice that the limit problem in the focusing case is global only in the sub cubical case

Ideas from the proof

The following a priori estimate is crucial

A priori estimate

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It is derived from the conservation of energy

$$\mathcal{E}^\varepsilon(\psi^\varepsilon(t)) = \int dx |\psi^{\varepsilon\prime}(t,x)|^2 + \frac{1}{\mu+1} \int_{\mathbb{R}} dx\, \frac{1}{\varepsilon} V\left(\frac{x}{\varepsilon}\right) |\psi^\varepsilon(t,x)|^{2\mu+2}$$

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Notice that it implies a uniform bound on the L^∞ norm

Ideas from the proof

Since the limit evolution has the form

$$\psi(t,x) = (U(t) * \psi_0)(x) - i\gamma \int_0^t U(t-s,x)|\psi(s,0)|^{2\mu}\psi(s,0)ds$$

the first step is the convergence of $\psi^\varepsilon(t,0)$

Convergence in the defect

Take $V \in L^1(\mathbb{R}, \langle x \rangle\, dx) \cap L^\infty(\mathbb{R})$ and let $V \geq 0$ or $\mu \in (0,1)$ then for any $T > 0$ and $0 < \delta < 1/2$ we have

$$\sup_{t\in(0,T)} |\psi^\varepsilon(t,0) - \psi(t,0)| \leq c\, \varepsilon^\delta$$

Ideas from the proof

As intermediate step we prove convergence in $L^2(\mathbb{R})$

L^2-convergence

Take $V \in L^1(\mathbb{R}, \langle x \rangle\, dx) \cap L^\infty(\mathbb{R})$ and let $V \geq 0$ or $\mu \in (0,1)$ then for any $T > 0$ and $0 < \delta < 1/2$ we have

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The convergence is strengthened to H^1 by a soft argument but we lose the rate.

Remarks

Some remarks

- the proof holds for N defects not just one
- nonlocal approximations are also possible
- notice that we assume the positivity of V not of $\gamma = \int V$
- we could soften the hypothesis on V

3d linear case

Delta-interaction in dimension three

$$H_{\alpha} = -\Delta + \alpha\delta_0$$

$$\mathcal{D}(H_{\alpha}) = \left\{ \psi = \phi + \frac{q}{4\pi|\mathbf{x}|};\ \phi \in \dot{H}^2(\mathbb{R}^3);\ q \in \mathbb{C};\ \phi(\mathbf{0}) = \alpha q \right\}$$

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$$H_{\alpha}\psi = -\Delta\phi \qquad \mathbf{x} \neq \mathbf{0}$$

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The evolution can be represented by

$$\psi(t,\mathbf{x}) = (U(t) * \psi_0)(\mathbf{x}) + i\int_0^t U(t-s,\mathbf{x})q(s)ds$$

$$q(t) + 4\sqrt{\pi i}\int_0^t \frac{\alpha q(s)}{\sqrt{t-s}}ds = 4\sqrt{\pi i}\int_0^t \frac{(U(s) * \psi_0)(0)}{\sqrt{t-s}}ds$$

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Theorem

Let $\psi_0 \in \mathcal{D}$, if $\gamma > 0$ and $\forall \mu > 0$ or if $\gamma < 0$ and $0 < \mu < 1$ then there is a a global solution $\psi \in C([0,T],\mathcal{D}) \cap C^1([0.T].\, L^2(\mathbb{R}^3))$. Moreover energy is conserved along the solutions.

Approximating problem in the linear case

The approximation of a three dimensional delta-interaction is more delicate than the one dimensional case.

$$-\Delta + \frac{1}{\varepsilon^3} V\left(\frac{x}{\varepsilon}\right) \not\to -\Delta + \alpha\delta_0$$

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More subtle phenomena are involved: resonant potential for local approximation, renormalization for non local approximation.

$$H_\varepsilon = -\Delta - \beta^\varepsilon |\rho_\varepsilon\rangle\langle\rho^\varepsilon| \qquad \rho^\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^3}\rho\left(\frac{\mathbf{x}}{\varepsilon}\right)$$

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$$\frac{1}{\beta^\varepsilon} = \frac{c_\rho}{\varepsilon} + \alpha \qquad c_\rho = \int d\mathbf{k}\, \frac{\hat{\rho}^2(\mathbf{k})}{|\mathbf{k}|^2}$$

It relies on a cancellation in an essential way.

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The linear case suggest as non local approximation:

$$i\frac{d}{dt}\psi^{\varepsilon}(t) = -\Delta\psi^{\varepsilon} + \frac{\varepsilon}{c_{\rho}}\left(-1+\gamma\frac{\varepsilon^{2\mu+1}|(\rho^{\varepsilon},\psi^{\varepsilon}(t))|^{2\mu}}{c_{\rho}^{2\mu+1}}\right)\frac{\varepsilon}{c_{\rho}}(\rho^{\varepsilon},\psi^{\varepsilon}(t))\rho^{\varepsilon}$$

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On the contrary, the solutions of the limit equation belongs to $\mathcal{D}$ which has trivial intersection with $H^2(\mathbb{R}^3)$.
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If we want to work with H^2-solutions of the approximating problem, we need a smooth initial datum:

$$\psi_0 = \phi_0 + \frac{q_0}{4\pi|\mathbf{x}|} \qquad \psi_0^\varepsilon = \phi_0 + \frac{q_0}{4\pi}\left(\rho^\varepsilon * \frac{1}{|\cdot|}\right)(\mathbf{x})$$

Main theorem

Theorem (Working form)

Let $\psi_0 \in \mathcal{D}$ and assume $\gamma > 0$ or $\gamma < 0$ and $0 < \mu < 1$ then for any $T > 0$ we have

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|\psi^{\varepsilon}(t) - \psi(t)\|_{L^2}$$

The L^2-convergence is reduced to proving that

$$\left| \int_0^t ds \frac{q^{\varepsilon}(s) - q(s)}{\sqrt{t-s}} \right| \to 0$$

while in the 1d case we essentially proved $\sup_t |q^{\varepsilon}(t) - q(t)| \to 0$

Perspectives

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- 3d nonlocal on form domain: space-time norm

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- 3d local approximation with resonant potential

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Some perspectives:

- 3d nonlocal on form domain: space-time norm
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- 1d local approximation with singular scaling of resonant potential