Time decay of the wave functions of magnetic Schrödinger operators

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General setting

Let $B : \mathbb{R}^2 \to \mathbb{R}$ be a magnetic field and consider the Schrödinger operator $H(B)$ in $L^2(\mathbb{R}^2)$ formally given by

$$H(B) = (i\nabla + A)^2$$

where $A : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $|A| \in L^2_{loc}(\mathbb{R}^2)$ and curl $A = B$ holds in the distributional sense.

- We will work under the condition $|A| \in L^\infty(\mathbb{R}^2)$; hence we define $H(B)$ as the unique self-adjoint operator associated with the closed quadratic form

$$Q[u] = \int_{\mathbb{R}^2} |(i\nabla + A)\, u|^2\, dx, \qquad d(Q) = W^{1,2}(\mathbb{R}^2).$$

General setting

Obviously, $H(B) \geq 0$. We assume that B is such that

$$\sigma(H(B)) = [0, \infty).$$

Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a bounded electric potential with a suitable decay at infinity such that $\sigma_{es}(H(B) + V) = [0, \infty)$.

■ The problem: we want to study the influence of the magnetic on the asymptotic behavior of the solutions to the Schrödinger equation

$$i\,\partial_t u = (H(B) + V)\,u$$

General setting

Hence the object our interest is the unitary group $e^{-it(H(B)+V)}$

In particular, we want to compare the time decay of

$$e^{-it(H(B)+V)}\, P_c^B \qquad \text{as} \qquad t \to +\infty$$

where P_c^B is the projection onto the continuous subspace of $L^2(\mathbb{R}^2)$ with respect to $H(B) + V$, with the decay of its non-magnetic counterpart:

$$e^{-it(-\Delta+V)}\, P_c \qquad \text{as} \qquad t \to +\infty$$

Here P_c is the projection onto the continuous subspace of $L^2(\mathbb{R}^2)$ with respect to $-\Delta + V$

Time decay: non-magnetic Schrödinger operators

- $L^1 \to L^\infty$ estimates: one considers the propagator $e^{-it(-\Delta+V)}\, P_c$ as operator from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ and studies the time decay of the corresponding norm

$$\|e^{-it(-\Delta+V)}\, P_c\|_{L^1\to L^\infty}$$

- If $V=0$, then

$$e^{it\Delta}(x,y) = (4\, i\, \pi\, t)^{-n/2}\, e^{\frac{i\,|x-y|^2}{4t}}\,, \qquad\qquad x,y\in\mathbb{R}^n$$

Hence

$$\|e^{it\Delta}\|_{L^1\to L^\infty} \;\leq\; (4\pi t)^{-\frac{n}{2}} \qquad t>0.$$

Time decay: non-magnetic Schrödinger operators

An alternative, thought less precise, way to measure the time decay is to consider $e^{-it(-\Delta+V)}$ as an operator between weighted L^2-spaces;

$$e^{-it(-\Delta+V)}\, P_c \;:\; L^2(\mathbb{R}^n, \rho^2\, dx) \to L^2(\mathbb{R}^n, \rho^{-2}\, dx),$$

or equivalently

$$\rho^{-1}\; e^{-it(-\Delta+V)}\, P_c\; \rho^{-1} \;:\; L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

where $\rho > 0$ is a suitable weight function.

For $V = 0$ the Cauchy-Schwarz inequality gives

$$\|\, \rho^{-1}\; e^{it\Delta}\; \rho^{-1}\; u\, \|_{L^2(\mathbb{R}^n)} \;\lesssim\; t^{-\frac{n}{2}}\; \|\rho^{-1}\,\|^2_{L^2(\mathbb{R}^n)}\; \|u\|_{L^2(\mathbb{R}^n)}$$

provided

$$\rho(x) = (1 + |x|)^{\frac{n}{2}+\varepsilon}, \quad \varepsilon > 0.$$

Time decay: non-magnetic Schrödinger operators

- If $V \neq 0$, then the decay rate depends on the validity of the estimate

$$\limsup_{z \to 0} \| \, \rho^{-1} \, (-\Delta + V - z)^{-1} \, \rho^{-1} \|_{2 \to 2} \; < \; \infty \tag{1}$$

If (1) holds true for some ρ, then we say that zero is a regular point of $-\Delta + V$; (generic situation).

- Zero is not a regular point of $-\Delta$ in $L^2(\mathbb{R}^n)$ for $n = 1, 2$.

- Zero is a regular point of $-\Delta$ in $L^2(\mathbb{R}^n)$ for $n \geq 3$:

$$\limsup_{z \to 0} \| \, \rho^{-1} \, (-\Delta - z)^{-1} \, \rho^{-1} \|_{2 \to 2} \; < \; \infty$$

if $\rho(x) = (1 + |x|)^{\beta}$, with $\beta \geq 1$.

Time decay: non-magnetic Schrödinger operators

- Dimension $n = 3$. If zero is a regular point of $-\Delta + V$, then as $t \to \infty$

$$\| \rho^{-1} e^{-it(-\Delta+V)} P_c \rho^{-1} \|_{2\to 2} = \mathcal{O}\big(t^{-\frac{3}{2}}\big) \qquad (2)$$

[Rauch 1978]: $\rho(x) = e^{\varepsilon|x|}$ and $V(x) \lesssim e^{-\varepsilon|x|}$, $\varepsilon > 0$.

[Jensen-Kato 1979]: $\rho(x) = (1 + |x|)^{\beta}$, $\beta > 5/2$, and $V(x) \lesssim (1 + |x|)^{-3}$.

[Journeé-Soffer-Sogge 1991, Goldberg-Schlag 2004, Goldberg 2006] ....

If zero is not a regular point of $-\Delta + V$, then (2) fails and one observes a slower decay: **[Rauch 1978, Jensen-Kato 1979, Murata 1982]**

Time decay: non-magnetic Schrödinger operators

- Dimension $n = 2$. **[Schlag 2005]** : if zero is a regular point of $-\Delta + V$, then

$$\| \rho^{-1} e^{-it(-\Delta+V)} P_c \, \rho^{-1} \|_{2\to 2} \; = \; \mathcal{O}(t^{-1}) \qquad t \to \infty. \tag{3}$$

holds for $\rho(x) = (1+|x|)^\beta, \ \beta > 1$ and $V(x) \lesssim (1+|x|)^{-3}$. This is again the decay rate of the free evolution. However, (3) **can be improved**, still under the condition that zero is a regular point, provided ρ grows fast enough:

$$\| \rho^{-1} e^{-it(-\Delta+V)} P_c \, \rho^{-1} \|_{2\to 2} \; = \; \mathcal{O}(t^{-1} (\log t)^{-2}) \qquad t \to \infty \tag{4}$$

where $\rho(x) = (1+|x|)^\beta, \ \beta > 3$, and $V(x) \lesssim (1+|x|^2)^{-3}$, **[Murata 82]**, see also **[Goldberg-Green 2013]**.

- Hence adding a potential V might improve the decay rate, contrary to the case $n \geq 3$.

Time decay: magnetic Schrödinger operators

- Dimension $n = 3$. **[Murata, 1982]** showed, under suitable regularity and decay assumptions on B and V, that if zero is a regular point of $H(B) + V$, and $\rho(x) = (1 + |x|)^\beta$ with β large enough, then

$$\| \, \rho^{-1} \, e^{-it(H(B)+V)} \, P_c \, \rho^{-1} \, \|_{2\to 2} = \mathcal{O}(t^{-3/2}) \qquad t \to \infty \tag{5}$$

and that the decay rate is sharp. Hence a magnetic field, sufficiently regular and decaying fast enough at infinity, **does not improve the decay rate** of $e^{-it(H(B)+V)}$ in dimension three.

- Dimension $n = 2$. Our motivation is to show that a compactly supported magnetic field in dimension two **does improve** the decay of $e^{-it(H(B)+V)}$ as $t \to \infty$ and that the decay rate is given by its **total flux**.

Main results: weighted L^2-estimates

- Assumption 1: Let $B \in C^\infty(\mathbb{R}^2; \mathbb{R})$ be such that for some $\sigma > 4$ we have

$$\sup_{\theta\in(0,2\pi)} \big(|B(r,\theta)| + |\,\partial_\theta B(r,\theta)|\big) \;\lesssim\; (1+r)^{-\sigma}.$$

Under this assumption we can define the following quantities:

$$\alpha = \frac{1}{2\pi} \int_{\mathbb{R}^2} B(x)\, dx \;<\; \infty, \qquad \mu(\alpha) := \min_{k\in\mathbb{Z}} |k-\alpha| \in [0, 1/2]\,.$$

- Assumption 2: Let $V : \mathbb{R}^2 \to \mathbb{R}$ be bounded and such that the operator $H(B) + V$ has no positive eigenvalues.

- $\sigma_{es}(H(B)+V) = \sigma_c(H(B)+V) = [0, \infty)$.

Main results: weighted L^2-estimates

Theorem (K.): Let $\alpha \notin \mathbb{Z}$. Put $\rho(x) = (1+|x|)^s$ with $s > 5/2$ and suppose that $|V(x)| \lesssim (1+|x|)^{-3}$. If zero is a regular point of $H(B)+V$, then there exists an operator

$$K(B,V) \in \mathscr{B}(L^2(\mathbb{R}^2))$$

such that

$$\rho^{-1}\, e^{-it(H(B)+V)}\, P_c^B\, \rho^{-1} \; = \; t^{-1-\mu(\alpha)}\, K(B,V) + o(t^{-1-\mu(\alpha)})$$

in $\mathscr{B}(L^2(\mathbb{R}^2))$ as $t \to \infty$.

Main results: weighted L^2-estimates

- The maximal decay rate $t^{-3/2}$, for $\mu(\alpha) = 1/2$, is the same as in dimension three.

- The operator $K(B, V)$ can be expressed explicitly in terms of B and V. Its L^2-norm is gauge-invariant.

- If $\rho(x) = (1 + |x|)^\beta$ then we must have $\beta \geq 1$.

- If $V = 0$, then zero is a regular point of $H(B)$:

$$\frac{1}{1 + |x|^2} \lesssim H(B)$$

in the sense of quadratic forms on $W^{1,2}(\mathbb{R}^2)$; **[Laptev-Weidl 1999]**.

Main results: weighted L^2-estimates

Theorem (K.): Let $\alpha \in \mathbb{Z}$. Put $\rho(x) = (1+|x|)^s$ with $s > 5/2$ and suppose that $|V(x)| \lesssim (1+|x|)^{-3}$. If zero is a regular point of $H(B)+V$, then there exists an operator

$$\widetilde{K}(B,V) \in \mathscr{B}(L^2(\mathbb{R}^2))$$

such that

$$\rho^{-1}\, e^{-it(H(B)+V)}\, P_c^B\, \rho^{-1} \; = \; t^{-1}(\log t)^{-2}\, \widetilde{K}(B,V) + o(t^{-1}(\log t)^{-2})$$

in $\mathscr{B}(L^2(\mathbb{R}^2))$ as $t \to \infty$.

Main ingredients of the proof

Assume that $\alpha \notin \mathbb{Z}$ and that $V = 0$.

By the spectral theorem and Stone formula we have

$$\rho^{-1}\, e^{-itH(B)}\, \rho^{-1} \, = \int_0^\infty e^{-it\lambda}\, E(\alpha,\lambda)\, d\lambda, \tag{6}$$

where $E(\alpha, \lambda)$ is the (weighted) spectral density associated to $H(B)$:

$$E(\alpha,\lambda) = \frac{1}{2\pi i} \lim_{\varepsilon\to 0+} \rho^{-1} \left[(H(B)-\lambda-i\varepsilon)^{-1} - (H(B)-\lambda+i\varepsilon)^{-1}\right] \rho^{-1}$$

We will use the notation

$$R_+(\alpha,\lambda) = \lim_{\varepsilon\to 0+} (H(B)-\lambda-i\varepsilon)^{-1}$$

Main ingredients of the proof

Let $\phi \in C^\infty(0,\infty)$, $0 \leq \phi \leq 1$, be such that $\phi(x) = 0$ for x large enough and $\phi(x) = 1$ in a neighborhood of 0.

$$\int_0^\infty e^{-it\lambda} E(\alpha,\lambda)\, d\lambda = \int_0^\infty e^{-it\lambda} (1-\phi)\, E(\alpha,\lambda)\, d\lambda + \int_0^\infty e^{-it\lambda} \phi\, E(\alpha,\lambda)\, d\lambda$$

Our aim is to show that

$$\int_0^\infty e^{-it\lambda} (1-\phi(\lambda))\, E(\alpha,\lambda)\, d\lambda \;=\; o(t^{-2})$$

and

$$\int_0^\infty e^{-it\lambda} \phi(\lambda)\, E(\alpha,\lambda)\, d\lambda \;=\; t^{-1-\mu(\alpha)} K(B,V) + o(t^{-1-\mu(\alpha)})$$

in $\mathscr{B}(L^2(\mathbb{R}^2))$ as $t \to \infty$.

Main ingredients of the proof

We need to prove that

$$E(\alpha, \lambda) = E_1 \, \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \qquad \lambda \to 0$$

for some $E_1 \in \mathscr{B}(L^2(\mathbb{R}^2))$. We have to show that

$$\rho^{-1} \, R_+(\alpha, \lambda) \, \rho^{-1} \; = \; F_0 + F_1 \, \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \qquad \lambda \to 0.$$

- Recall that in the absence of magnetic field we have

$$\rho^{-1} \, R_+(\lambda) \, \rho^{-1} \; = \widetilde{F}_0 \, \log \lambda + \mathcal{O}(1) \qquad \lambda \to 0.$$

Resolvent expansion at threshold

Consider a radial magnetic field B_0 generated by the vector potential

$$A_0(x) = \alpha\,(-x_2, x_1) \begin{cases} |x|^{-1} & |x| \leq 1 \\ |x|^{-2} & |x| > 1 \end{cases} \qquad \nabla \cdot A_0 = 0.$$

$$B_0(x) = \operatorname{curl} A_0(x) = \begin{cases} \alpha\,|x|^{-1} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}, \qquad \frac{1}{2\pi} \int_{\mathbb{R}^2} B_0(x)\,dx = \alpha.$$

Using the partial wave decomposition, after some calculations we find that

$$\rho^{-1}\; R^0_+(\alpha, \lambda)\; \rho^{-1} \;=\; G_0 + G_1\,\lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \quad \lambda \to 0$$

for some G_0, G_1 in $\mathscr{B}(L^2(\mathbb{R}^2))$, where $R^0_+(\alpha, \lambda)$ is the resolvent of $H(B_0)$.

Resolvent expansion at threshold

Lemma: Let $\alpha > 0$ be the flux of B through $\mathbb{R}^2$. Then there exists a bounded vector field $A = (a_1, a_2)$ s. t. curl $A = \partial_1 a_2 - \partial_2 a_1 = B$ in the distributional sense, and

$$|\nabla \cdot A(x)| = o\left(|x|^{-3}\right), \qquad |A(x) - A_0(x)| = o\left(|x|^{-3}\right)$$

- The above Lemma implies that

$$T(B) := H(B) - H(B_0) = 2i \underbrace{(A - A_0)}_{o\left(|x|^{-3}\right)} \cdot \nabla + \underbrace{i \nabla \cdot A}_{o\left(|x|^{-3}\right)} + \underbrace{|A|^2 - |A_0|^2}_{o\left(|x|^{-3}\right)}$$

since $\nabla \cdot A_0 = 0$. This allows us to show that the operator

$$G_0\, \rho\, T(B)\, \rho = \rho^{-1} H(B_0)^{-1} T(B)\, \rho$$

is compact in $\mathscr{B}(L^2(\mathbb{R}^2))$.

Resolvent expansion at threshold

With this we prove that $1 + G_0\, \rho\, T(B)\, \rho$ is invertible in $L^2(\mathbb{R}^2)$. Then

$$1 + \rho^{-1} R_+^0(\alpha, \lambda)\; T(B)\, \rho = 1 + G_0\; \rho\, T(B)\, \rho\; + G_1\; \rho\, T(B)\, \rho\; \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)})$$

is invertible for λ small enough. From the resolvent equation we thus obtain

$$\rho^{-1}\; R_+(\alpha, \lambda)\; \rho^{-1} = \left(1 + \rho^{-1} R_+^0(\alpha, \lambda)\; T(B)\, \rho\right)^{-1} \rho^{-1}\; R_+^0(\alpha, \lambda)\; \rho^{-1}$$

Since

$$\left(1 + \rho^{-1} R_+^0(\alpha, \lambda)\; T(B)\, \rho\right)^{-1} = (1 + G_0\; \rho\, T(B)\, \rho)^{-1} + S(B)\, \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}),$$

we arrive at

$$\rho^{-1}\; R_+(\alpha, \lambda)\; \rho^{-1} \;=\; F_0 + F_1\, \lambda^{\mu(\alpha)} + o(\lambda^{\mu(\alpha)}) \qquad \lambda \to 0.$$

Remark

- In order that the coefficients of $H(B_1, V) - H(B_2, V)$ decay faster than $o(|x|^{-1})$ at infinity, the fluxes of B_1 and B_2 must be equal.

Indeed, if $\operatorname{curl} A_1 = B_1$ and $\operatorname{curl} A_2 = B_2$, then by the Stokes Theorem we have

$$|A_1(x) - A_2(x)| = o(|x|^{-1}) \quad |x| \to \infty \quad \Rightarrow \quad \int_{\mathbb{R}^2} B_1(x)\, dx = \int_{\mathbb{R}^2} B_2(x)\, dx.$$

$L^1 \to L^\infty$ estimates: Aharonov-Bohm field

Let $V = 0$ and let B_{ab} be given by Aharovon-Bohm magnetic field with flux α:

$$A_{ab}(x) = (A_1(x), A_2(x)) = \frac{\alpha}{|x|^2}\,(-x_2\,,\,x_1) \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\},$$

We define the $H(B_{ab}) =: H_\alpha$ in $L^2(\mathbb{R}^2)$ as the Friedrichs extension of

$$(i\nabla + A_{ab})^2 \qquad \text{on} \quad C_0^\infty(\mathbb{R}^2 \setminus \{0\})$$

Recently is was proved by **[Fanelli-Felli-Fontelos-Primo 2013]** that

$$\|e^{-itH_\alpha}\|_{L^1 \to L^\infty} \lesssim \frac{1}{t} \qquad \forall\, t > 0.$$

$L^1 \to L^\infty$ estimates: Aharonov-Bohm field

Theorem (G.Grillo, H.K.): we have

$$\| \, \rho^{-1} \, e^{-itH_\alpha} \, \rho^{-1} \, \|_{L^1 \to L^\infty} \; = \; \mathcal{O}\left(t^{-1-\mu(\alpha)}\right) \qquad t \to \infty$$

where

$$\rho(x) = (1 + |x|)^{\mu(\alpha)}.$$

The proof of this result is based on the explicit knowledge of $e^{-itH_\alpha}(x, y)$:

$$e^{-i\,tH_\alpha}(x, y) = \frac{1}{4\pi i\,t}\, e^{-\frac{r^2+r'^2}{4it}} \sum_{m\in\mathbb{Z}} I_{|m+\alpha|}\left(\frac{rr'}{2it}\right) e^{im(\theta-\theta')}\,,$$

where $I_\nu(\cdot)$ is the modified Bessel function of order ν and

$$x_1 + ix_2 = re^{i\theta}, \quad y_1 + iy_2 = r'e^{i\theta'}.$$

$L^1 \to L^\infty$ estimates: Aharonov-Bohm field

We use the estimate

$$\sup_{x,y\in\mathbb{R}^2} \left| e^{-i\,tH_\alpha}(x,y) \right| \lesssim t^{-1}$$

established in [Fanelli-Felli-Fontelos-Primo 2013] to prove that

$$\sup_{x,y\in\mathbb{R}^2} \left| (1+|x|)^{-\mu(\alpha)}\, e^{-i\,tH_\alpha}(x,y)\, (1+|y|)^{-\mu(\alpha)} \right| \lesssim t^{-1-\mu(\alpha)}$$

From here the claim follows immediately.

$L^1 \to L^\infty$ estimates: Aharonov-Bohm field

- The growth of ρ in the estimate

$$\| \, \rho^{-1} \, e^{-itH_\alpha} \, \rho^{-1} \, \|_{L^1 \to L^\infty} \; = \; \mathcal{O}\left(t^{-1-\mu(\alpha)}\right) \qquad t \to \infty \tag{7}$$

cannot be improved.

- For $\alpha \in \mathbb{Z}$ equation (7) turns into

$$\| \, e^{-itH_\alpha} \, \|_{L^1 \to L^\infty} \; = \; \mathcal{O}(t^{-1}) \qquad t \to \infty$$

which is the day rate of the free evolution; $H_\alpha \simeq -\Delta$ if $\alpha \in \mathbb{Z}$.