Controllability of spin-boson systems

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Control of a quantum system

Consider the Schrödinger equation

$$\begin{aligned} i\dot{\psi} &= \left(H_0 + \sum_{k=0}^{m} u_k H_k \right) \psi, & \psi \in \mathcal{H} \\ \psi(0) &= \psi_i & \psi_i \in \mathcal{H} \end{aligned} \tag{1}$$

where:

i) H_k, $k = 1, ..., m$ are selfadjoint operators on $\mathcal{H}$ called control operators

ii) $u_k = u_k(t) \in U \subset \mathbb{R}$, $k = 1, ..., m$ are real valued functions called control functions. These functions belongs to a subset $\mathfrak{U}$ of a suitable functional space (usually $\mathcal{PC}(\mathbb{R}^+; \mathcal{U})$).

Problem

Given a final state $\psi_f \in \mathcal{H}$, find a finite time $T > 0$ and control functions $u_k : [0, T] \to U$, $k = 1, ..., m$ such that $\psi(t)$, solution to (1), satisfies $\psi(T) = \psi_f$. If such controls could be founded we will say that ψ_f is reachable from ψ_i.

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Questions and aims

1 Which points of the space are reachable from ψ_i?
$\hookrightarrow$ Determine the set of states reachable from a given ψ_i.

Definition

The system (1) is controllable if any final state $\psi_f \in \mathcal{H}$ is reachable from any initial state $\psi_i \in \mathcal{H}$.

2 Is it possible to find explicitly the control functions?
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Finite dimensional quantum systems

If $dim\mathcal{H} = n < \infty$ one could determine the controllability properties of the systems looking at the propagator dynamic

$$\begin{aligned} i\dot{U} &= \left(H_0 + \sum_{k=0}^{m} u_k H_k \right) U, \quad U \in U(n) \\ U(0) &= \mathbb{1} \end{aligned} \tag{2}$$

Theorem

The set of states reachable from the identity for system (2) is given by the connected Lie subgroup $e^{\mathcal{L}}$, corresponding to the Lie algebra $\mathcal{L}$, generated by $\{iH_0, iH_1, ..., iH_m\}$.
System (2) is operator-controllable if and only if $\mathcal{L} = \mathfrak{u}(n)$ or $\mathcal{L} = \mathfrak{su}(n)$.

Example: Two level system

$$i\dot{\psi} = \frac{\hbar\omega}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi + u(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi \tag{3}$$

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Infinite dimensional framework

$dim\mathcal{H} = \infty$. Consider the following system with one control

$$\begin{cases} i\dot{\psi} = (H_0 + uH_1)\psi \\ \psi(0) = \psi_i \end{cases} \quad (4)$$

where H_0 and H_1 are selfadjoint operators, u is a piecewise constant function with value in $U \subset \mathbb{R}$. Moreover suppose that the following assumptions holds:

Assumption 1

- u1) exists an orthonormal basis $\Phi = \{\phi_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ consisting of eigenfunctions of H_0. For every $n \in \mathbb{N}$ $H_0\phi_n = \lambda_n\phi_n$;
- u2) for every $n \in \mathbb{N}$, ϕ_n is in $D(H_1)$, the domain of H_1;
- u3) for every $u \in U$, $(H_0 + uH_1)$ is essentially self adjoint on $Span\{\phi_n\}_{n\in\mathbb{N}}$;
- u4) if $\lambda_j = \lambda_k$ and $j \neq k$ then $\langle \phi_j, H_1\phi_k \rangle = 0$.

If $u = \sum_j u_j \mathbb{1}_{[t_{j+1}-t_j]}$, a solution to (4) could be written as

$$\psi(t) = \Upsilon^u_t(\psi_i) = e^{(t-t_k)(H_0+u_kH_1)} e^{t_{k-1}(H_0+u_{k-1}H_1)} \circ \cdots \circ e^{t_1(H_0+u_1H_1)}\psi_i$$

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A weak notion of controllability

In general for an infinite dimensional quantum system the reachable set from a given ψ_0 is dense in $D(H_0)$ ([BMS][T]). So the notion of exact controllability is too strong for such a system.

Example: Square potential well

$$i\dot{\psi} = -\frac{1}{2}\Delta\psi + ux\psi, \qquad x \in (0,1)$$

The attainable set with L^2 controls from the first eigenstate of the Laplacian is

$$L^2((0,1),\mathbb{C}) \cup \{\psi \in H^3((0,1),\mathbb{C}) \mid \psi(0) = \psi(1) = \psi(0) = \psi(1) = 0\}$$

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Definition

Let (H_0, H_1, U, Φ) satisfy Assumption 1. We say that (4) is approximately controllable if for every $\varepsilon > 0$, for every ψ_i, ψ_f in the unit sphere of $\mathcal{H}$, there exists a piecewise constant function $u_\varepsilon : [0, T_\varepsilon] \to U$ such that

$$\| \Upsilon^{u_\varepsilon}_{T_\varepsilon}(\psi_i) - \psi_f \| < \varepsilon$$

Example: Harmonic oscillator

$$i\dot{\psi} = \frac{1}{2}(-\Delta + x^2)\psi + ux\psi, \qquad x \in \mathbb{R}$$

This system is not controllable in any reasonable sense [MR].

$$\frac{d}{dt}\langle x\rangle_\psi = \langle p\rangle_\psi, \quad \frac{d}{dt}\langle p\rangle_\psi = -\langle x\rangle_\psi + u$$

$$i\frac{d}{dt}\varphi = \frac{1}{2}(\tilde{p}^2 + z^2)\varphi \qquad \left(z = x - \langle x\rangle, \tilde{P} = \partial/\partial z\right)$$

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A sufficient condition for approximate controllability

Consider a subset $S \subset \mathbb{N}^2$ that satisfies the following conditions

Non-resonance: for every $(j,k) \in S$

$$|\lambda_j - \lambda_k| \neq |\lambda_s - \lambda_t| \qquad \forall (s,t) \in \mathbb{N}^2 \setminus \{(j,k),(k,j)\} \text{ s. t. } \langle B\phi_s, \phi_t \rangle \neq 0$$

Connectedness by chains: for every $(j,k) \in \mathbb{N}^2$ exists a finite sequence $\{s_1, ..., s_m\} \subset \mathbb{N}$ such that

i) $s_1 = j$ and $s_m = k$;

ii) $(s_i, s_{i+1}) \in S$ for $i = 1, ..., m-1$;

iii) $\langle \phi_{s_i}, H_1 \phi_{s_{i+1}} \rangle \neq 0$ for $i = 1, ..., m-1$.

We will say that a subset with these properties is a *non-resonant connectedness chain* for the system (4).

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Theorem (Boscain et al. [BCCS])

Let $\delta > 0$ and let $(H_0, H_1, [0,\delta], \Phi)$ satisfy Assumption 1. If there exists a non-resonant connectedness chain for $(H_0, H_1, [0,\delta], \Phi)$ then (4) is approximately controllable.

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$$i\dot{\psi} = -\frac{1}{2}\Delta\psi + ux\psi, \qquad x \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad u \in [0,\delta]$$

Could be proof that the equivalent system

$$i\dot{\psi} = A_\eta\psi + vx\psi, \qquad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

where $A_\eta = \frac{1}{2}\Delta + \eta x$, is approximately controllable with $v \in [\eta, \delta - \eta]$.

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Spin-boson systems

Consider $L^2(\mathbb{R};\mathbb{C})\otimes\mathbb{C}^2$, the Hilbert space of a bosonic mode of a quantized field in interaction with a two-level system. The free hamiltonian for this system is

$$h_0 = \frac{\hbar\omega}{2}\left(-\frac{d^2}{dx^2}+x^2\right)\otimes\mathbb{1}+\frac{\hbar\Omega}{2}\mathbb{1}\otimes\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \hbar\omega\left(a^\dagger a+\frac{1}{2}\right)\otimes\mathbb{1}+\frac{\hbar\Omega}{2}\mathbb{1}\otimes\sigma_z$$

Different interaction terms gives different models

$$H_{Rabi}(g) = h_0 + gx\otimes\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = h_0 + g(a^\dagger + a)\sigma_x$$

$$H_{JC}(g) = h_0+\frac{g}{\sqrt{2}}\left[x\otimes\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+i\frac{d}{dx}\otimes\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = h_0+g(a^\dagger\sigma_-+a\sigma_+)$$

The constant $g\in\mathbb{R}$ determines the strong of the interaction.

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Spin-boson systems

We will study the controllability of these system for the control operator $H_1 = x \otimes \mathbb{1}$, an external potential corresponding to a constant electric field.

$$i\dot{\psi}(t) = (H_0 + u(t)H_1)\psi(t) \quad \text{with} \quad H_0 = H_{Rabi}(g)/H_{JC}(g) \tag{5}$$

We observe that the Assumption 1 hold.

- h_0 has a complete set of eigenvectors $\{\Phi_{n,s} \mid n \in \mathbb{N}, s \in \{-1,1\}\}$ of eigenvalues $E_{n,s} = \hbar\omega(n+1/2) + \hbar\Omega s/2$
- $H_{Rabi}(g)$ and $H_{JC}(g)$ are small perturbations of h_0 (in sense of Kato). Then, they have a complete set of eigenvectors for every $g \in \mathbb{R}$ (by the Kato-Rellich theorem).

$$g \mapsto \Phi_{n,s}(g) = \Phi_{n,s}(0) + \sum_{m=1}^{\infty} \Phi_{n,s}^{(m)} g^m \qquad g \mapsto E_{n,s}(g) = E_{n,s}(0) + \sum_{m=1}^{\infty} E_{n,s}^{(m)} g^m$$

- H_1 is Kato small with respect to h_0, so with respect to H_0
- $(u4)$ holds for almost any $g \in \mathbb{R}$.

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- $(u4)$ holds for almost any $g \in \mathbb{R}$.

Non resonance of the spectrum

In order to apply Theorem 1 we have to study the resonances of the eigenvalues. Consider the set

$$Q = \{(\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}) \in (\mathbb{N} \times \{-1, 1\})^4 \mid (\mathbf{i}, \mathbf{j}) \neq (\mathbf{k}, \mathbf{l}) \quad \text{e} \quad \mathbf{i} \neq \mathbf{j}\}$$

Take an element $(\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l})$. By the analytic dependence on g of the eigenvalues to prove that

$$E_{\mathbf{i}}(g) - E_{\mathbf{j}}(g) \neq E_{\mathbf{k}}(g) - E_{\mathbf{l}}(g) \quad \text{for a.e. g}$$

it is sufficient to prove that the Taylor expansions of $E_{\mathbf{i}} - E_{\mathbf{j}}$ and $E_{\mathbf{k}} - E_{\mathbf{l}}$ in zero are different.

Imposing that

$$E_{\mathbf{i}}(0) - E_{\mathbf{j}}(0) = E_{\mathbf{k}}(0) - E_{\mathbf{l}}(0)$$
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$$E_{\mathbf{i}}^{(4)}(0) - E_{\mathbf{j}}^{(4)}(0) = E_{\mathbf{k}}^{(4)}(0) - E_{\mathbf{l}}^{(4)}(0)$$

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$$\omega \notin \mathbb{N}\Omega$$

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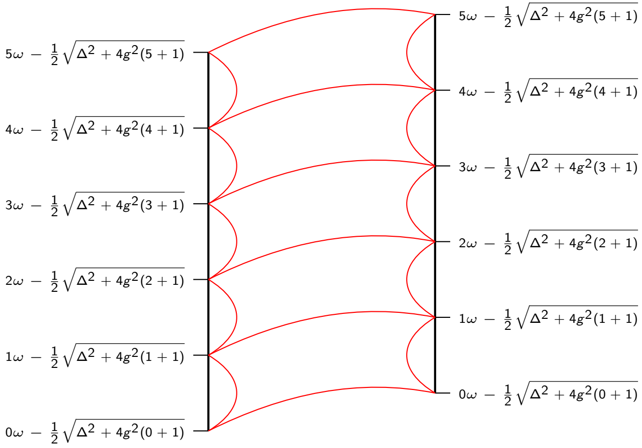
Connectedness chain

Consider the set

$$S = \left\{ (\mathbf{m}, \mathbf{n}) \in (\mathbb{N} \times \{-1, 1\})^2 \quad \middle| \quad \frac{1}{2} |s(\mathbf{m}) - s(\mathbf{n})| + |n(\mathbf{m}) - n(\mathbf{n})| = 1 \right\}$$



$$\Delta = \omega - \Omega$$

Connectedness chain

It remains to prove that

$$\langle \Phi_{\mathbf{j}}(g)\,,\,(x \otimes \mathbb{1})\Phi_{\mathbf{k}}(g)\rangle \neq 0$$

for a.e. $g \in \mathbb{R}$ and for all $(\mathbf{j}, \mathbf{k}) \in S$. This is similar to the previous calculations.

Theorem

Assume that ω is not an integer multiple of Ω. Then system (5), with $H_1 = x \otimes \mathbb{1}$ is approximately controllable for almost every $g \in \mathbb{R}$.

Assume that $\omega = \Omega$. Then the system (5) is approximately controllable for almost every $g \in \mathbb{R}$.

Connectedness chain

It remains to prove that

$$\langle \Phi_{\mathbf{j}}(g) \, , \, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}(g) \rangle \neq 0$$

for a.e. $g \in \mathbb{R}$ and for all $(\mathbf{j}, \mathbf{k}) \in S$. This is similar to the previous calculations.

Theorem

Assume that ω is not an integer multiple of Ω. Then system (5), with $H_1 = x \otimes \mathbb{1}$ is approximately controllable for almost every $g \in \mathbb{R}$.
Assume that $\omega = \Omega$. Then the system (5) is approximately controllable for almost every $g \in \mathbb{R}$.

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