NLS ground states on graphs

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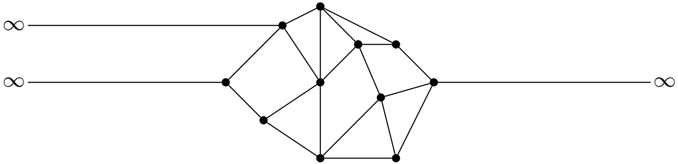
TQMS, Como, 8–10/7/2015

The problem

Let $\mathcal{G}$ be a noncompact metric graph



Consider the functional $E(\,\cdot\,,\mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}$ defined by

$$E(u,\mathcal{G}) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\,dx$$

The problem

Take $\mu > 0$ and define

$$H^1_\mu(\mathcal{G}) = \{u \in H^1(\mathcal{G}) \,:\, \|u\|^2_{L^2(\mathcal{G})} = \mu\}.$$

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Critical points of $E(u, \mathcal{G})$ on $H^1_\mu(\mathcal{G})$ satisfy

- there exists $\omega \in \mathbb{R}$ such that on every edge

$$u'' + |u|^{p-2}u = \omega u \qquad \text{(NLS equation)}$$

- for every vertex v

$$\sum_{e \succ \mathrm{v}} \frac{du_e}{dx_e}(\mathrm{v}) = 0 \qquad \text{(Kirchhoff conditions)}$$

The problem

Given a non–compact graph $\mathcal{G}$ we look for critical points of E on $H^1_\mu(\mathcal{G})$, starting from the simplest type:

global minimizers, or ground states of mass μ

Set

$$\mathcal{E}_{\mathcal{G}}(\mu) = \inf_{v \in H^1_\mu(\mathcal{G})} E(v, \mathcal{G}).$$

Thus we are looking for functions $u \in H^1_\mu(\mathcal{G})$ such that

$$\boxed{E(u, \mathcal{G}) = \mathcal{E}_{\mathcal{G}}(\mu).}$$

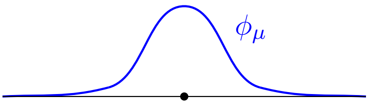
In particular, we want to understand how the topology of $\mathcal{G}$ influences the existence of ground states.

Examples

1. The real line $(\mathcal{G} = \mathbb{R})$

For $p \in (2,6)$ and $\mu > 0$ ground states exist and are the family of translates of the soliton

$$\phi_\mu(x) \;=\; C\mu^{\frac{2}{6-p}}\, sech^{\frac{2}{p-2}}\big(c\mu^{\frac{p-2}{6-p}}x\big).$$



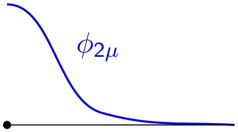
Their level $\mathcal{E}_{\mathbb{R}}(\mu)$ plays a very important role in what follows.

When $p = 4$, for example,

$$\phi_\mu(x) \;= \frac{\mu}{2\sqrt{2}} sech\left(\frac{\mu}{4}x\right), \qquad \mathcal{E}_{\mathbb{R}}(\mu) = -\frac{\mu^3}{96}, \qquad \omega = \frac{\mu^2}{16}.$$

2. The half–line $(\mathcal{G} = \mathbb{R}^+)$

For $p \in (2, 6)$ and $\mu > 0$ there is exactly one ground state given by "half a soliton" of mass 2μ.

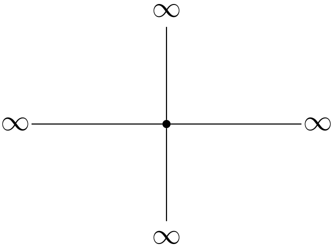


When $p = 4$,

$$\phi_{2\mu}(x) = \frac{\mu}{\sqrt{2}} sech\left(\frac{\mu}{2}x\right), \qquad \mathcal{E}_{\mathbb{R}^+}(\mu) = -\frac{\mu^3}{24}, \qquad \omega = \frac{\mu^2}{4}.$$

3. Infinite *n*–star graphs

(Adami-Cacciapuoti-Finco-Noja '12)

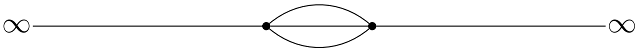


For $p \in (2,6)$ and $\mu > 0$,

$$\inf_{u \in H^1_\mu(\mathcal{G})} E(u,\mathcal{G}) = \mathcal{E}_{\mathbb{R}}(\mu)$$

but the infimum is not achieved: there is no ground state.

4. Bridges



For $p \in (2,6)$ and $\mu > 0$,

$$\inf_{u \in H^1_\mu(\mathcal{G})} E(u,\mathcal{G}) = \mathcal{E}_{\mathbb{R}}(\mu)$$

and again the infimum is not achieved: there is no ground state.

This conclusion is independent of the number of bridges connecting the two vertices. It is very simple to prove if the number of bridges is odd, and highly nontrivial if it is even.

Framework

From now on

- $\mathcal{G}$ is a generic non–compact graph (contains at least one half–line)
- the mass $\mu > 0$ is fixed
- $E(\,\cdot\,, \mathcal{G}) : H^1_\mu(\mathcal{G}) \to \mathbb{R}$ is defined by

$$E(u, \mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\,dx$$

- $p \in (2, 6)$ (this makes E bounded below for every μ)

General results

Theorem (Level pinching)

For every non–compact graph $\mathcal{G}$,

$$\mathcal{E}_{\mathbb{R}^+}(\mu) \le \inf_{u \in H^1_\mu(\mathcal{G})} E(u,\mathcal{G}) \le \mathcal{E}_{\mathbb{R}}(\mu)$$

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Theorem (Existence)

For every non–compact graph $\mathcal{G}$, *if*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) < \mathcal{E}_{\mathbb{R}}(\mu),$$

then the infimum is attained, namely $\mathcal{G}$ *supports a ground state.*

General results

We are now going to see how the topology of the graph affects the existence of ground states.
Consider the following assumption.

(H) Every $x \in \mathcal{G}$ lies on a *trail* that contains two half-lines

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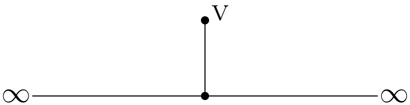
- No graph with only one half-line can satisfy (H)
- (H) is violated also by the presence of terminal edges



General results

Theorem (Nonexistence)

Assume that $\mathcal{G}$ satisfies assumption (H). Then

$$\inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) = \mathcal{E}_{\mathbb{R}}(\mu)$$

and it is never attained, except if $\mathcal{G}$ is a "tower of bubbles".

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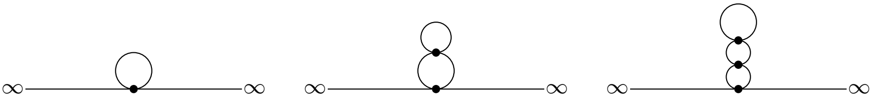

Some towers of bubbles

Sketch of the proof that (H) $\Longrightarrow$ nonexistence.

Let $u \in H^1_\mu(\mathcal{G})$, and let x_0 be a global maximum point for u.

Take a trail $\mathcal{T}$ through x_0 that contains two half-lines.

Then u restricted to $\mathcal{T}$ is in $H^1(\mathcal{T})$ and $\max_{\mathcal{T}} u = \max_{\mathcal{G}} u$.

$$\#\{x \in \mathcal{G} \,:\, u(x) = t\} \geq \#\{x \in \mathcal{T} \,:\, u(x) = t\} \geq 2 \quad \text{for } a.e.\ t$$

Therefore, if $\widehat{u}$ is the symmetric rearrangement of u on $\mathbb{R}$,

$$\begin{aligned} E(u,\mathcal{G}) &= \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\,dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}} |\widehat{u}'|^2\,dx - \frac{1}{p}\int_{\mathbb{R}} |\widehat{u}|^p\,dx \geq \mathcal{E}_{\mathbb{R}}(\mu). \end{aligned}$$

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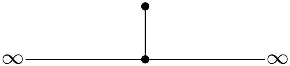
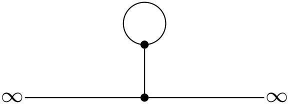
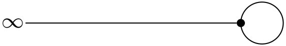


A line with a terminal edge



A signpost graph



A tadpole graph

A 3–fork graph

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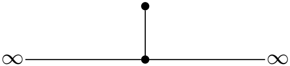

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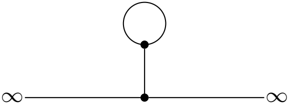

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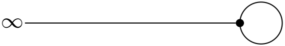

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For each of these, we can show that the infimum is attained.

The technique is by rearrangements, to produce functions u such that $E(u, \mathcal{G}) < \mathcal{E}_{\mathbb{R}}(\mu)$. By the existence theorem, the conclusion follows.

Part 2: the critical case $p = 6$

Now we turn to the problem of the existence of minimizers for

$$E(u,\mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{6}\int_{\mathcal{G}} |u|^6\,dx$$

on $H^1_\mu(\mathcal{G})$.

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on $H^1_\mu(\mathcal{G})$.

These are solutions of the L^2–critical stationary NLS equation

$$u'' + u^5 = \omega u \qquad \text{on } \mathcal{G},$$

with Kirchhoff boundary conditions.

This problem is much more delicate than the subcritical one.

One of the reasons is that under the formal mass-preserving transformation

$$u(x) \mapsto u_\lambda(x) = \sqrt{\lambda}u(\lambda x),$$

the kinetic and the potential terms in E scale in the same way:

$$E(u_\lambda, \lambda^{-1}\mathcal{G}) = \lambda^2 E(u, \mathcal{G}),$$

which is typical of problems with serious loss of compactness.

In the critical case the problem depends very strongly on μ.

The real line ($\mathcal{G} = \mathbb{R}$)

It is known that there exists a number $\mu_{\mathbb{R}} > 0$, the critical mass, such that

$$\mathcal{E}_{\mathbb{R}}(\mu) = \begin{cases} -\infty & \text{if } \mu > \mu_{\mathbb{R}} \\ 0 & \text{if } \mu \le \mu_{\mathbb{R}} \end{cases} \qquad \left(\mu_{\mathbb{R}} = \pi\sqrt{3}/2\right).$$

Moreover $\mathcal{E}_{\mathbb{R}}(\mu)$ is attained if and only if $\mu = \mu_{\mathbb{R}}$.

The ground states form a quite large family: up to sign and translations, they can be written as

$$\phi_\lambda(x) = \sqrt{\lambda}\phi(\lambda x), \qquad \lambda > 0,$$

where $\phi(x) = sech^{1/2}(\frac{2}{\sqrt{3}}x)$.

Interpretation of the critical mass $\mu_{\mathbb{R}}$

The best constant in the Gagliardo–Nirenberg inequality

$$\|u\|_6^6 \le C\|u\|_2^4\,\|u'\|_2^2 \qquad \forall u \in H^1(\mathbb{R})$$

is

$$K_{\mathbb{R}} = \sup_{\substack{u\in H^1(\mathbb{R})\\ u\not\equiv 0}} \frac{\|u\|_6^6}{\|u\|_2^4\,\|u'\|_2^2}.$$

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$$K_{\mathbb{R}} = \sup_{\substack{u \in H^1(\mathbb{R}) \\ u \not\equiv 0}} \frac{\|u\|_6^6}{\|u\|_2^4 \, \|u'\|_2^2}.$$

Then, for every $u \in H^1_\mu(\mathbb{R})$,

$$E(u, \mathbb{R}) = \frac{1}{6} \left(3\|u'\|_2^2 - \|u\|_6^6 \right) \ge \frac{1}{6} \|u'\|_2^2 \left(3 - K_{\mathbb{R}} \mu^2 \right)$$

If $\mu^2 < 3/K_{\mathbb{R}}$, then $E(u, \mathbb{R}) > 0$, and (by scaling u appropriately),

$$\mathcal{E}_{\mathbb{R}}(\mu) = \inf_{u \in H^1_\mu(\mathbb{R})} E(u, \mathbb{R}) = 0.$$

On the other hand, if $\mu^2 > 3/K_{\mathbb{R}}$, and u is close to optimality in the Gagliardo–Nirenberg inequality,

$$E(u,\mathbb{R}) \leq \frac{1}{6}\|u'\|_2^2 \left(3 - (K_{\mathbb{R}} - \varepsilon)\mu^2\right) < 0,$$

and, again by scaling,

$$\mathcal{E}_{\mathbb{R}}(\mu) = \inf_{u \in H^1_\mu(\mathbb{R})} E(u,\mathbb{R}) = -\infty.$$

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Then we see that

$$\mu_{\mathbb{R}}^2 = \frac{3}{K_{\mathbb{R}}}.$$

On a generic noncompact graph $\mathcal{G}$, it is therefore natural to define the critical mass as

$$\mu_{\mathcal{G}}^2 = \frac{3}{K_{\mathcal{G}}},$$

where $K_{\mathcal{G}}$ is the best constant for the Gagliardo–Nirenberg inequality on $\mathcal{G}$.

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Remark. It is easy to see that for every noncompact $\mathcal{G}$,

$$K_{\mathbb{R}} \leq K_{\mathcal{G}} \leq K_{\mathbb{R}^+}$$

so that

$$\mu_{\mathbb{R}^+} \leq \mu_{\mathcal{G}} \leq \mu_{\mathbb{R}}.$$

Repeating the argument shown for $\mathcal{G} = \mathbb{R}$ one sees immediately that

$$\mu > \mu_{\mathcal{G}} \quad \Longrightarrow \quad \mathcal{E}_{\mathcal{G}}(\mu) < 0 \quad (\text{possibly } -\infty)$$

$$\mu \leq \mu_{\mathcal{G}} \quad \Longrightarrow \quad \mathcal{E}_{\mathcal{G}}(\mu) = 0$$

Remark. It is important to keep in mind that for $\mathcal{G} = \mathbb{R}$ or $\mathcal{G} = \mathbb{R}^+$, the value of the mass that ensures the existence of a ground state is unique.

One of the points of interest of our work is that this property is no longer true on certain graphs. In other words, in some cases ground states exist for a whole interval of masses.

The location of the critical mass $\mu_{\mathcal{G}}$ in the interval $[\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$, the value of $\mathcal{E}_{\mathcal{G}}(\mu)$ and the existence of ground states depend on the topology of the graph.

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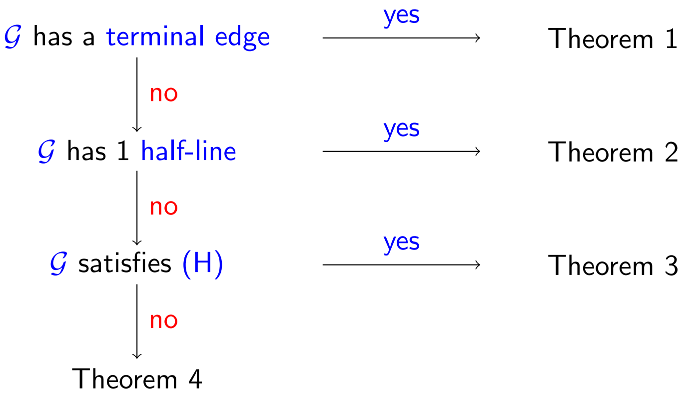
We will treat various cases, according to the following scheme.



Theorem (1)

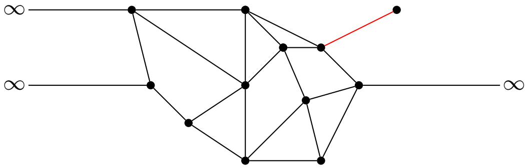
Let $\mathcal{G}$ be a noncompact graph having at least one terminal edge. Then

$$\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$$

and

$$\mathcal{E}_{\mathcal{G}}(\mu) = \begin{cases} -\infty & \text{if} \quad \mu > \mu_{\mathbb{R}^+} \\ 0 & \text{if} \quad \mu \leq \mu_{\mathbb{R}^+}. \end{cases}$$

A ground state exists if and only if $\mu = \mu_{\mathbb{R}^+}$ and $\mathcal{G}$ is a half-line.



Theorem (2)

Let $\mathcal{G}$ be a noncompact graph having exactly one half-line and no terminal edge. Then

$$\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$$

and

$$\mathcal{E}_{\mathcal{G}}(\mu) \begin{cases} = -\infty & \text{if} \quad \mu > \mu_{\mathbb{R}} \\ < 0 & \text{if} \quad \mu_{\mathbb{R}^+} < \mu \le \mu_{\mathbb{R}} \\ = 0 & \text{if} \quad \mu \le \mu_{\mathbb{R}^+}. \end{cases}$$

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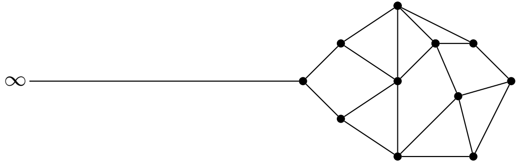

Theorem (3)

Let $\mathcal{G}$ be a noncompact graph satisfying assumption (H). Then

$$\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$$

and

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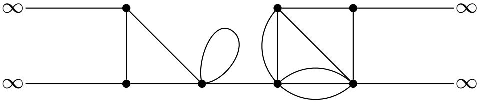
A ground state exists if and only if $\mu = \mu_{\mathbb{R}}$ and $\mathcal{G}$ is a tower of bubbles.



Theorem (4)

There exist noncompact graphs $\mathcal{G}$, without terminal edges, with more than one half-line and that do not satisfy assumption (H), such that

$$\mu_{\mathbb{R}^+} < \mu_{\mathcal{G}} < \mu_{\mathbb{R}}$$

and

$$\mathcal{E}_{\mathcal{G}}(\mu) \begin{cases} = -\infty & \text{if} \quad \mu > \mu_{\mathbb{R}} \\ < 0 & \text{if} \quad \mu_{\mathcal{G}} < \mu \leq \mu_{\mathbb{R}} \\ = 0 & \text{if} \quad \mu \leq \mu_{\mathcal{G}}. \end{cases}$$

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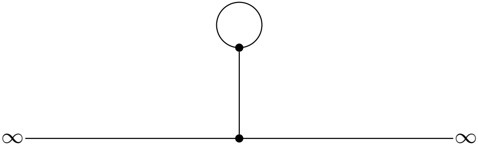

Comments.

- In theorems 1 and 3 the situation is similar to that of $\mathbb{R}$: a ground state exists for a single value of the mass and moreover the graph is forced to have a particular structure (half-line, tower of bubbles...).
- On the other hand theorems 2 and 4 describe a completely new phenomenon: ground states exist for all values of μ in a nontrivial interval. This is due to the different topology of certain graphs with respect to that of $\mathbb{R}$.

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We now sketch a "cumulative" proof of (part of) theorems 2 and 4.

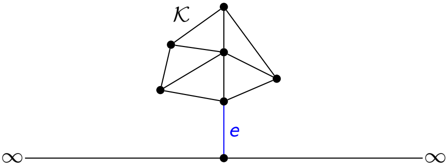
Proof (ideas) in the case $\mu_{\mathcal{G}} < \mu < \mu_{\mathbb{R}}$, so that $\mathcal{E}_{\mathcal{G}}(\mu) < 0$.

First we obtain bounds for a minimizing sequence u_n in the relevant norms.

Then we pass to the limit in $E(u_n, \mathcal{G})$.

Bounds

1. Since $\mathcal{G}$ does not satisfy (H), there exists a cut-edge e and a compact connected component $\mathcal{K}$ of $\mathcal{G} \setminus e$.



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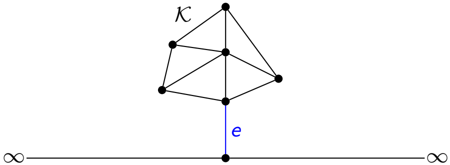
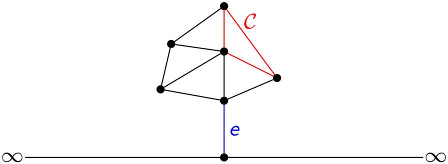


2. $\mathcal{K}$ cannot be a vertex, otherwise e would be a terminal edge, excluded by assumption.

3. $\mathcal{K}$ must contain a cycle $\mathcal{C}$, otherwise $\mathcal{K}$ would be a tree, and have a terminal edge.



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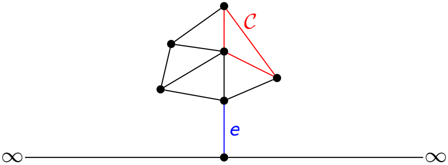

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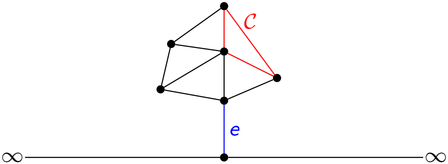
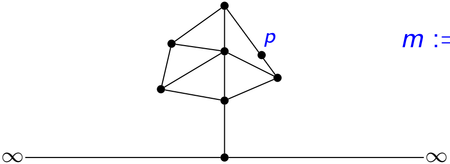


4. From now on we assume that the cut-edge e is the only one.

5. For every $u \in H^1_\mu(\mathcal{G})$, let p be an absolute minimum point for u on $\mathcal{C}$, and let $m = u(p)$.



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6. Note that $m^2|\mathcal{C}| \leq \int_{\mathcal{C}} |u|^2\,dx \leq \int_{\mathcal{G}} |u|^2\,dx = \mu$, so

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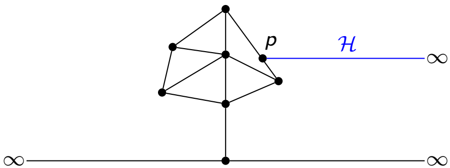
7. We attach a half-line $\mathcal{H}$ to $\mathcal{G}$ at p, and call $\mathcal{G}'$ the new graph, that now satisfies (H).



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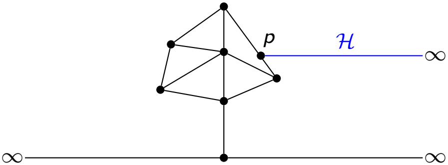

8. We extend u to $\mathcal{G}'$ by setting

$$w(x) = \begin{cases} u(x) & \text{if} \quad x \in \mathcal{G} \\ me^{-x/2\varepsilon} & \text{if} \quad x \in \mathcal{H} \end{cases}$$

9. Clearly $w \in H^1(\mathcal{G}')$ and

$$\|w\|_2^2 = \mu + \varepsilon m^2 \leq \mu(1 + \varepsilon c)$$

$$\|w'\|_2^2 = \|u'\|_2^2 + \frac{m^2}{\varepsilon} \leq \|u'\|_2^2 + \frac{c\mu}{\varepsilon}$$

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$$= 3\frac{\mu^2}{\mu_{\mathbb{R}}^2}\,(1+\varepsilon c)^2\left(\|u'\|_2^2 + \frac{c\mu}{\varepsilon}\right).$$

Thus,

$$\|u'\|_2^2 \le \frac{\mu^2}{\mu_{\mathbb{R}}^2}(1+\varepsilon c)^2\left(\|u'\|_2^2 + \frac{\mu}{\varepsilon}c\right)$$

for every $u \in H^1_\mu(\mathcal{G})$ such that $E(u,\mathcal{G}) < 0$.

Let

$$\theta = \frac{\mu^2}{\mu_{\mathbb{R}}^2}(1+\varepsilon c)^2$$

and note that $0 < \theta < 1$ if ε is chosen small.

Then

$$(1-\theta)\|u'\|_2^2 \le \theta\frac{c\mu}{\varepsilon}.$$

Conclusion: there exists $C > 0$ such that

$$\begin{cases} u \in H^1_\mu(\mathcal{G}) \\ \\ E(u,\mathcal{G}) < 0 \end{cases} \qquad \Longrightarrow \qquad \|u'\|_2 \le C.$$

It is then easy to obtain further estimates like

$$\|u\|_6 \le C, \quad \|u\|_\infty \le C, \ldots .$$

from which we also see that

$$\mathcal{E}_{\mathcal{G}}(\mu) > -\infty.$$

Limit

Let $u_n \in H^1_\mu(\mathcal{G})$ be a minimizing sequence:

$$E(u_n, \mathcal{G}) \to \mathcal{E}_{\mathcal{G}}(\mu) < 0.$$

By the preceding estimates we can assume that

$$u_n \rightharpoonup u \quad \text{in } H^1(\mathcal{G})$$

$$u_n \to u \quad \text{in } L^q_{\mathrm{loc}}(\mathcal{G}) \quad \forall q \in [1 + \infty]$$

$$u_n(x) \to u(x) \quad a.e.$$

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5. Hence,

$$E(u_n, \mathcal{G}) \geq E(u, \mathcal{G}) + o(1),$$

that is,

$$E(u, \mathcal{G}) \leq \mathcal{E}_{\mathcal{G}}(\mu).$$

Conclusion. If u has mass $m < \mu$, then the function $v = \sqrt{\mu/m}\, u$ has mass μ and

$$E(v, \mathcal{G}) < \frac{\mu}{m} E(u, \mathcal{G}) < \mathcal{E}_{\mathcal{G}}(u),$$

a contradiction.
Therefore u has mass μ and is the required ground state.