NLS equation on metric graphs with localized nonlinearities

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joint work with Enrico Serra

TQMS – Como 09/07/2015

Some background

Applications:

- the study of the NLS equation on graphs arises in complex networks, quantum chaotic systems and Bose–Einstein condensates (Gnutzmann, Smilansky - 2006);
- the issue of localized nonlinearity is of interest in transmission through complex networks of one–dimensional leads, e.g. optical fiber (Gnutzmann, Smilansky, Derevyanko - 2011).

Mathematical background:

- NLS with localized nonlinearities in standard domains (different point of view): Adami, Dell'Antonio, Figari, Teta – 2003; Cacciapuoti, Finco, Noja, Teta – 2014;
- NLS with non-localized nonlinearities on graphs: Adami, Cacciapuoti, Finco, Noja – 2012, 2014; Adami, Serra, Tilli – 2014, 2015.

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Basics on metric graphs

As usual, with (connected) metric graph we mean a multigraph

$$\mathcal{G} = (\mathrm{V}, \mathrm{E})$$

(possibly multiple edges and self-loops), where each edge e joining two vertices v_1 and v_2 is associated either with a closed bounded interval

$$I_e = [0, \ell_e] \subset \mathbb{R} \quad \Rightarrow \quad e \text{ is a bounded edge}$$

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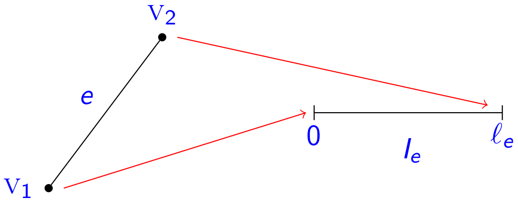
Moreover, a coordinate x_e is chosen in I_e, so that v_1 corresponds to $x_e = 0$ and v_2 to $x_e = \ell_e$



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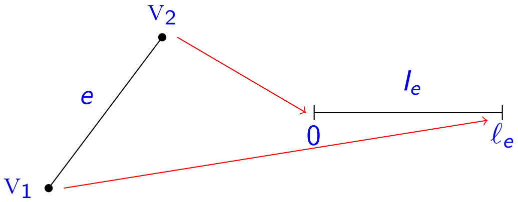

If $\ell_e = +\infty$, we always assume that the half-line I_e is attached to the graph at $x_e = 0$, and the vertex corresponding to $x_e = +\infty$ is called a vertex at infinity.

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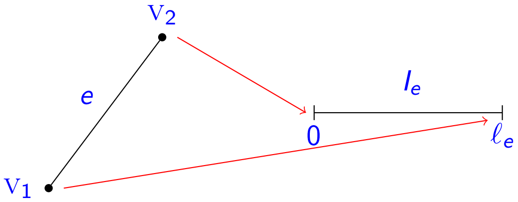

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Compact core

We say that a metric graph $\mathcal{G}$ is compact if it does not contain any unbounded edge (or, equivalently, any vertex at infinity).

Definition

The *compact core* $\mathcal{K}$ is the metric subgraph of $\mathcal{G}$ consisting of all the bounded edges (and the related vertices).

Note that:

- $\mathcal{K}$ is a compact metric graph;
- $\mathcal{K}$ can be empty;
- if $\mathcal{G}$ is compact (and non–trivial), then $\mathcal{K} = \mathcal{G}$.

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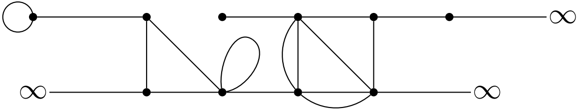

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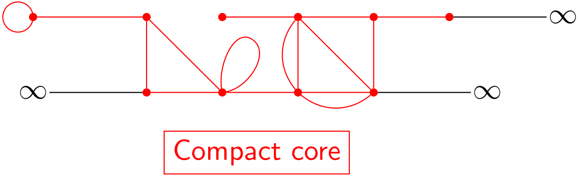

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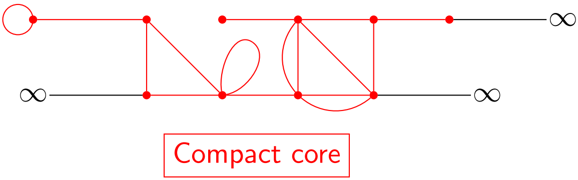

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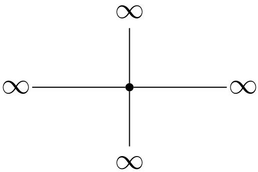

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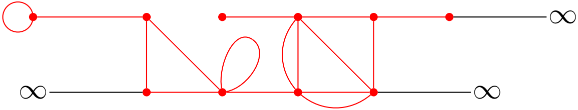

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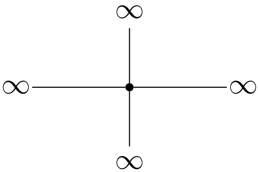

Compact core $\Rightarrow$ $\emptyset$

Functions on graphs

A function $u : \mathcal{G} \to \mathbb{R}$ is a family of functions $u = (u_e)_{e\in\mathrm{E}}$, with

$$u_e : I_e \to \mathbb{R} \qquad \forall e \in \mathrm{E}.$$

The usual Lebesgue function spaces can be defined as

$$u \in L^p(\mathcal{G}) \Leftrightarrow u_e \in L^p(I_e) \quad \forall e \in \mathrm{E}, \qquad \|u\|^p_{L^p(\mathcal{G})} = \sum_{e\in\mathrm{E}} \|u_e\|^p_{L^p(I_e)},$$

while $H^1(\mathcal{G})$ is the set of continuous $u = (u_e)_e$ s.t.

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Bound states of prescribed mass

Let $\mathcal{G}$ be a metric graph with nonempty $\mathcal{K}$ and let $p \in [2, +\infty)$.

A *bound state of mass* $\mu > 0$ can be characterized as a function $u \in H^1(\mathcal{G})$, with $\|u\|^2_{L^2(\mathcal{G})} = \mu$, for which:

(i) there exists $\lambda \in \mathbb{R}$ s.t.

$$u_e'' + \chi_{\mathcal{K}}(x)|u_e|^{p-2}\,u_e = \lambda u_e \qquad \text{(stationary NLS equation);}$$

(ii) for every vertex v in $\mathcal{K}$

$$\sum_{e \succ \mathrm{v}} \frac{du_e}{dx_e}(\mathrm{v}) = 0 \qquad \text{(Kirchhoff condition).}$$

Note: $du_e/dx_e(\mathrm{v})$ stands for $u_e'(0)$ or $-u_e'(\ell_e)$ depending on the orientation of I_e.

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Connection with the NLS energy

Let $E : H^1(\mathcal{G}) \to \mathbb{R}$ be the functional

$$E(u) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{K})} = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{K}} |u|^p\,dx.$$

For fixed $\mu > 0$ define

$$M = \{u \in H^1(\mathcal{G}) : \|u\|^2_{L^2(\mathcal{G})} = \mu\}$$

and denote by E_M the restriction of E to M. Then, E_M is of class C^1 and critical points of E_M satisfy

$$\int_{\mathcal{G}} u'\varphi'\,dx - \int_{\mathcal{K}} |u|^{p-2}\,u\varphi\,dx + \lambda\int_{\mathcal{G}} u\varphi\,dx = 0 \quad \forall\varphi \in H^1(\mathcal{G}),$$

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Ground states of prescribed mass

Finally, we say that $u \in M$ is a *ground state of mass* $\mu > 0$ if

$$E_M(u) = \inf_{v \in M} E_M(v).$$

Remarks:

- u is a critical point of E_M and hence a bound state of mass μ (the one with the least energy level);
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A general result

Theorem – T. – 2015

Let $\mathcal{G}$ be a noncompact connected metric graph with nonempty compact core $\mathcal{K}$. Let $p \in (2,6)$ and $\mu > 0$. Then

$$\inf_{v \in M} E_M(v) \leq 0.$$

Moreover, if $\inf_{v \in M} E_M(v) < 0$, then the infimum is attained.

Note: the case $p \in (2,6)$ is usually called L^2–subcritical case.

Note: in the sequel we tacitly suppose that $\mathcal{G}$, $\mathcal{K}$, p and μ always satisfy the previous assumptions.

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Existence result

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- If $p \in (2, 4)$, for every $\mu > 0$ there exists a ground state of mass μ.
- If $p \in [4, 6)$, there exists $\mu_1 > 0$ s.t., for every $\mu > \mu_1$, there exists a ground state of mass μ.

Note: μ_1 (not sharp in general) satisfies

$$\mu_1^{\frac{p-2}{6-p}} L = c_p N^{\frac{4}{6-p}}$$

where $L = \mathrm{meas}(\mathcal{K})$, N is the number of unbounded edges of $\mathcal{G}$ and c_p is a positive constant.

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Sketch of the proof

For $\alpha \in (0, \sqrt{\mu/L})$ and $m = \frac{\mu - \alpha^2 L}{N}$, define the competitor

$$u(x) = \begin{cases} \alpha & \text{in } \mathcal{K} \\ \alpha e^{-\frac{\alpha^2 x}{2m}} & \text{in each half-line.} \end{cases}$$

Then $E_M(u)$ reads

$$E_M(u) = \frac{\alpha^4 N^2}{8(\mu - \alpha^2 L)} - \frac{\alpha^p L}{p}$$

If $p \in (2,4)$, then $E_M(u) < 0$ provided α is small. If $p \in [4,6)$, then a sufficient condition to have a value $\alpha_0 \in (0, \sqrt{\mu/L})$ s.t. $E_M(u) < 0$ is that $\mu > \mu_1$.

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If $p \in [4, 6)$, there exists $\mu_2 > 0$ s.t., for every $\mu < \mu_2$, there cannot exist any ground state of mass μ.

Note: also μ_2 can be explicitly computed and satisfies

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The threshold on μ in the existence result is an actual phenomenon ~~or a lack of our proof~~.

Theorem – T. – 2015

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Partition of a graph

The number of unbounded edges N affects existence since

$$\mu_1 = \left(c_p L^{-1} N^{\frac{4}{6-p}}\right)^{\frac{6-p}{p-2}}.$$

How is it that it does not affect nonexistence (recall that μ_2 does not depends on N)? $\Rightarrow$ Actually it does!

Indeed, if $N \geq 2$, then we can cover $\mathcal{G}$ with a family of metric graphs $(\mathcal{G}_i)_{i=1}^{\nu}$ $(2 \leq \nu \leq N)$ pairwise disjoint (up to sets of zero measure) and such that each $\mathcal{G}_i$ contains at least an unbounded edge. We call such a family a partition of $\mathcal{G}$.

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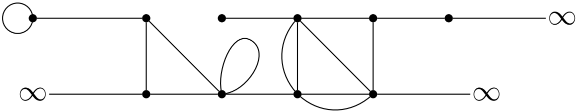

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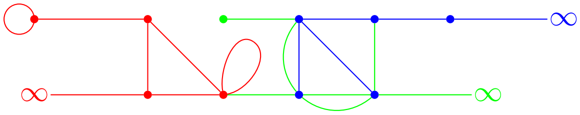

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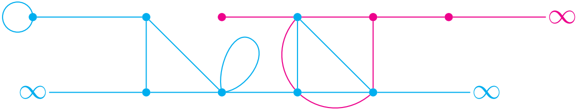

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Another one by $\mathcal{G}_4$, $\mathcal{G}_5$.

And so on ...

Improvement of nonexistence

Corollary – T. – 2015

Let $(\mathcal{G}_i)_{i=1}^{\nu}$ be a partition of $\mathcal{G}$. Setting $L_i = \mathrm{meas}(\mathcal{G}_i \cap \mathcal{K})$, if

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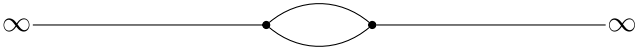

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If one choose partition $\mathcal{G}_1$, $\mathcal{G}_2$, then $L_1 = L_2 = L/2$ and $\widehat{\mu}_i > \mu_2$.

Basics on solitons

The soliton of mass μ is the unique minimizer (up to a change of sign and translations) of

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}} |u'|^2\,dx - \frac{1}{p}\int_{\mathbb{R}} |u|^p\,dx$$

and the unique solution of

$$u'' + |u|^{p-2}\,u = \lambda u$$

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$$\varphi_\mu(x) = \mu^{\frac{2}{6-p}}\varphi_1\left(\mu^{\frac{p-2}{6-p}}x\right)$$

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Existence of multiple bound states

Theorem – Serra, T. – 2015

For every $k \in \mathbb{N}$, there exists $\mu_k > 0$ such that for all $\mu \geq \mu_k$ there exist at least k distinct pairs $(\pm u_j)$ of bound states of mass μ. Moreover, for every $j = 1, \dots, k$

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where $\sigma_k(\mu) \to 0$ (exponentially fast) as $\mu \to \infty$. Finally, for each j, the Lagrange multiplier λ_j related to u_j is positive.

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Take, for istance $\mathcal{G}$

- In the degenerate case when the interval shrinks to a point ($\mathcal{K} = \emptyset$) the problem becomes linear $\Rightarrow$ there cannot exist any bound state of mass μ.
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Nonlinearity on a "compact portion of positive measure" generates bound states at higher energies!

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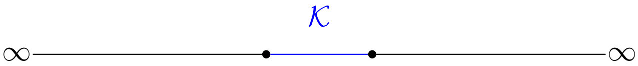

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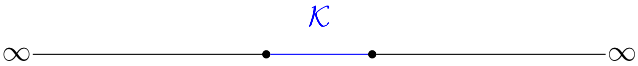

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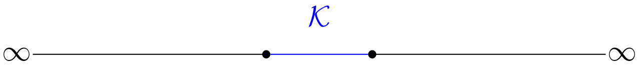

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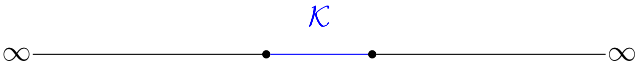

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Sketch of the proof: Minimax methods

For every $A \subset H^1(\mathcal{G})\setminus\{0\}$ closed and symmetric we define the *Krasnosel'skii genus* of A as

$$\gamma(A) = \min\{n \in \mathbb{N} : \exists f : A \to \mathbb{R}^n\setminus\{0\} \text{ continuous and odd}\}.$$

Then, introducing the minimax classes

$$\Gamma_j = \{A \subset M : A \text{ compact, symmetric and } \gamma(A) \geq j\},$$

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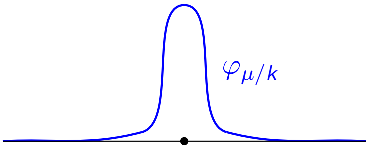

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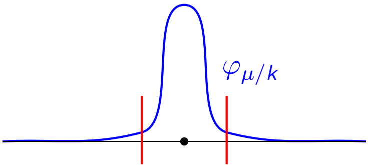

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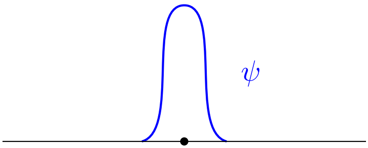

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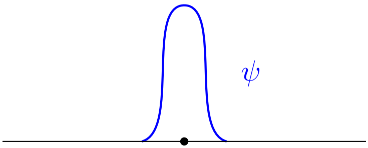

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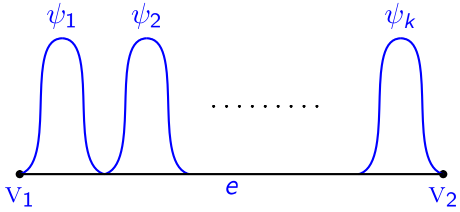

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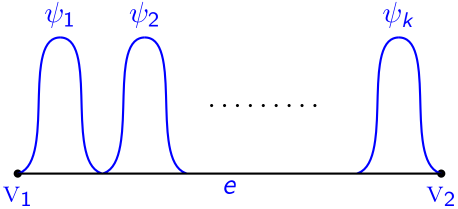

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Last question: where we used the fact that μ is large? why we need to enlarge μ when k increases?

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Beacause we must place more and more disjoint copies of ψ (with smaller and smaller support) on e keeping at the same time the energy level negative.

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And finally...

THANK YOU FOR YOUR ATTENTION!

Sketch of the proof: a general result

To prove the first part, one can easily find a sequence $(v_k) \subset M$ s.t. $\lim_k E_M(v_k) = 0$.

On the other hand, using the L^p–version of the Gagliardo-Nirenberg Inequality

$$\|u\|^p_{L^p(\mathcal{G})} \leq \mathcal{C}_p \|u\|^{\frac{p}{2}+1}_{L^2(\mathcal{G})} \|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})} \quad \forall u \in H^1(\mathcal{G}),$$

there results that each minimizing sequence $(u_k) \subset M$ is bounded in $H^1(\mathcal{G})$.

Then $u_k \rightharpoonup u$ in $H^1(\mathcal{G})$ and, by Rellich Theorem, we see that

$$E(u) \leq \liminf_k E_M(u_k).$$

Since $\inf_{v \in M} E_M(v) < 0$ prevents $\|u\|^2_{L^2(\mathcal{G})} < \mu$, one concludes.

Sketch of the proof: nonexistence result (1/2)

Since $\inf_{v\in M} E_M(v) \leq 0$ for all $\mu > 0$, it is sufficient to find $\mu_2 > 0$ s.t. for all $\mu < \mu_2$ and all $u \in M = \{u \in H^1(\mathcal{G}) : \|u\|^2_{L^2(\mathcal{G})} = \mu\}$ there results

$$E_M(u) > 0.$$

By an inductive argument one sees that $E_M(u) \leq 0$ entails

$$\|u'\|_{L^2(\mathcal{G})^2} \leq \frac{1}{\mathcal{C}_\infty^4 \mu} \|u\|_{L^\infty(\mathcal{G})}^{4(\frac{p}{4})^{n+1}} (\mathcal{C}_\infty^4 \mu L)^{\sum_{i=1}^n (\frac{p}{4})^i} \quad \forall n \geq 0,$$

by a repeated use of the L^∞–version of GNI

$$\|u\|_{L^\infty(\mathcal{G})} \leq \mathcal{C}_\infty \|u\|_{L^2(\mathcal{G})}^{1/2} \|u'\|_{L^2(\mathcal{G})}^{1/2} \quad \forall u \in H^1(\mathcal{G}).$$

Sketch of the proof: nonexistence result (2/2)

Then $\forall n \geq 0$

$$\|u'\|^2_{L^2(\mathcal{G})} \leq \mathcal{C}_p^2 \mu^3 (\mathcal{C}_\infty^4 \mu L)^{n+1} \qquad \text{if } p = 4$$

$$\|u'\|^2_{L^2(\mathcal{G})} \leq \mathcal{C}_p^{\frac{4}{6-p}} \mu^{\frac{p+2}{6-p}} \left(\mathcal{C}_\infty^{\frac{4p}{p-4}} \mathcal{C}_p^{\frac{4}{6-p}} \mu^{\frac{4(p-2)}{(p-4)(6-p)}} L^{\frac{4}{p-4}} \right)^{(\frac{p}{4})^{n+1}-1} \qquad \text{if } p > 4.$$

If the terms in brackets are < 1, that is

$$\mu < \mu_2 = \left(L^{-1} \mathcal{C}_p^{\frac{4-p}{6-p}} \mathcal{C}_\infty^{-p} \right)^{\frac{6-p}{p-2}},$$

then $\|u'\|^2_{L^2(\mathcal{G})} = 0$. Since $u \in H^1(\mathcal{G})$, there follows that $u \equiv 0$, but this is a contradiction with $\|u\|^2_{L^2(\mathcal{G})} = \mu > 0 \Rightarrow$ ~~$E_M(u) \leq 0$~~.

A remark on homothety

Let $u \in M$ and $\sigma > 0$. The function $w(x) = \sigma^{\frac{2}{6-p}} u\left(\sigma^{\frac{p-2}{6-p}} x\right)$ belongs to $M_\sigma = \{v \in H^1(\mathcal{G}_\sigma) : \|v\|^2_{L^2(\mathcal{G}_\sigma)} = \sigma\mu\}$, where $\mathcal{G}_\sigma$ is a metric graph obtained from $\mathcal{G}$ by an homothety of factor $\sigma^{-\frac{p-2}{6-p}}$. Moreover, $E_{M_\sigma}(w) = \sigma^{\frac{p+2}{6-p}} E_M(u)$ and hence

w is a ground state of mass $\sigma\mu$ in $\mathcal{G}_\sigma$

$$\Updownarrow$$

u is a ground state of mass μ in $\mathcal{G}$

Note that:

- μ_1 and μ_2 scale coherently with this transformation;
- one can formulate all the previous results in terms of L in place of μ.

Sketch of the proof: Palais-Smale condition

We say that $(u_k) \subset M$ is a *Palais-Smale sequence* at level c if

$$E_M(u_k) \to c \qquad \|E'_M(u_k)\| \to 0$$

and that E_M satisfies the *Palais-Smale condition* at level c (i.e., $(PS)_c$) if each PS sequence admits a converging subsequence in M.

It is possible to check that E_M satisfies $(PS)_c$ only when $c < 0$:

- counterexamples for $c \geq 0$;
- standard techniques for $c < 0$.